Chapter 1.5

Collisional Transport in Plasma

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Contents

Handbook of Plasma Physics, Eds. M.N. Rosenbluth and R.Z. Sagdeev
Volume 1: Basic Plasma Physics I, edited by A.A. Galeev and R.N. Sudan

1.5.1. Introduction

Transport processes in plasmas are responsible for the loss of particles and energy in situations where "ideal" plasmas would be expected to be perfectly confined, and for the departure from ideal behavior in other situations as well. Among the mechanisms for plasma transport are (1) macroscopic instabilities, (2) turbulence due to microinstabilities, and (3) Coulomb collisions. Only the last mechanism is considered in this chapter.

The force on a given charged particle due to the other particles in the plasma comes from the electric and magnetic field fluctuations. These fluctuations may be Fourier analyzed into plane-wave components with different wavenumbers $\boldsymbol{k}$. By "collisions" we mean the fluctuations with wavelengths shorter than a Debye length or $k\lambda_D > 1$.

For a system in local thermal equilibrium, the level of such fluctuations is well known (Thompson and Hubbard, 1960; Rostoker, 1961) and the effect on the one-particle distribution function is described by the Fokker–Planck equation, (1). This article treats only "collisional" transport and does not discuss transport due to longer-wavelength fluctuations. Not only is collisional transport much better understood, it is always present, and sets a lower limit on the rate of entropy production and transport in a plasma. It is therefore assumed, implicitly, that macroscopic MHD activity (or rapid macroscopic motion) is absent, and that microinstability-associated fluctuations are also absent.

In this chapter only transport in MHD equilibrium systems is considered, in which the current density and the pressure gradient are related by (161). The time rate of change of macroscopic quantities is assumed to be small, and due entirely to transport. Thus, the effects of transport on waves or instabilities are not considered, unless they are of sufficiently low frequency and long wavelength to satisfy the assumptions made in Section 1.5.3. Transport in more rapidly varying situations has been reviewed by Kaufman (1960) and Braginskii (1965). A review by Kadomtsev (1963) should also be noted.

Plasma transport properties are derived here using an expansion procedure due to Chapman and Enskog (Chapman and Cowling, 1952). The small parameter is taken to be the ratio of the mean free path to the characteristic length parallel to the magnetic field, which is assumed to be of the same order of magnitude as the ratio of the mean particle gyroradius to the perpendicular characteristic length. The fluxes of particles and energy are calculated only to first order in these small parameters. Third-order fluxes, related to viscous forces and higher-order thermal forces, are omitted. The time dependences are assumed to be of second order in the small parameter, compared with the rate of particle gyration in the magnetic field.

Although some general formulas are given, this chapter will be concerned mainly with transport in magnetically confined plasmas, which are strongly magnetized. This means that the mean particle gyroradius is assumed to be much smaller than the mean free path. More general results may be found in the paper by Shkarofsky et al. (1963). Explicit formulas for transport perpendicular to a strong magnetic field are given for a plasma with an arbitrary number of different particle species (ionic species distinguished by ionic charge and mass, in addition to electrons). Transport parallel to the magnetic field is discussed for a simple plasma, with electrons plus one ionic species, and also for a plasma with multiple ionic species.

In Section 1.5.2, the Fokker–Planck description of velocity space diffusion is introduced. Properties of the Fokker–Planck collision terms are derived first, then macroscopic relaxation times and the approach to thermal equilibrium are discussed. The motion of a test particle is described next, and some important applications are discussed. Finally, the approximations to the Fokker–Planck collision terms, which are used in this article, are derived.

In Section 1.5.3, the classical transport processes due to spatial gradients and electric and magnetic fields are discussed. The basic expansion procedure is described first, then transport perpendicular to a magnetic field, in the strong field limit, is derived. The next subsection 3.3 specializes to a simple plasma, indicates the changes to the foregoing theory when the smallness of the electron-to-ion mass ratio is exploited, and gives results for the parallel current density and the heat fluxes carried by electrons and ions. A variational formulation is used to demonstrate the Onsager symmetry relations. In the next subsection, transport in a plasma with multiple ion species is considered. The ratio of electron mass to any ion mass is assumed to be small, but the ratio of any two ion masses is arbitrary. In addition to the electron fluxes, general results for the parallel ion transport coefficients are presented. Finally, the form of the moment equations for the multiple ion-species plasma is given and all of the preceding results are summarized.

1.5.2. Velocity space diffusion

The Fokker–Planck equation

Plasma collisional transport is due to the fundamental graininess of the medium, which is manifested through Coulomb scattering of each of the discrete charges by all of the others. In calculating transport properties, one needs to know the effect of these charged-particle interactions on the one-particle distribution function $f(\boldsymbol{x}, \boldsymbol{v}, t)$. This function is defined such that

$$f(\boldsymbol{x}, \boldsymbol{v}, t)\, \mathrm{d}^3x\, \mathrm{d}^3v$$

is proportional to the probability of finding one particle of a given species in the phase-space volume element centered at $\boldsymbol{x}, \boldsymbol{v}$ whose infinitesimal volume is $\mathrm{d}^3x\, \mathrm{d}^3v$ at time t. This expression is equal to the statistical expectation value of the number of such particles in the phase-space volume element at time t. Since this chapter will not be directly concerned with fluctuations, $f(\boldsymbol{x}, \boldsymbol{v}, t)$ (which will be called the *distribution function*) will be thought of as the actual density of a continuous fluid in phase space. The spatial number density is then given by

$$n(\boldsymbol{x}, t) = \int \mathrm{d}^3v\, f(\boldsymbol{x}, \boldsymbol{v}, t).$$

For the most common applications of plasma transport theory, it is assumed that the effect of charged-particle interactions is adequately described by the Fokker–Planck equation (Rosenbluth et al., 1957). In this section this equation will be discussed, as well as some of the approximations used in deriving it, and some limitations on its applicability.

In a homogeneous plasma, in the absence of external electric and magnetic fields, the time dependence of the distribution functions is assumed to be determined by the Fokker–Planck equations (one equation for each particle species a):

$$\frac{\partial f_a}{\partial t} = -\frac{\partial}{\partial \boldsymbol{v}} \cdot \left[\boldsymbol{A}_a f_a - \frac{1}{2} \frac{\partial}{\partial \boldsymbol{v}} \cdot (\boldsymbol{D}_a f_a) \right] \tag{1}$$

where the dynamical friction vector is

$$\boldsymbol{A}_a = -2 \sum_b \frac{c_{ab}}{m_a^2} \left(1 + \frac{m_a}{m_b}\right) \int \mathrm{d}^3v' f_b(\boldsymbol{v}') \frac{\boldsymbol{u}}{u^3} \equiv \sum_b \boldsymbol{A}_{ab} \tag{2}$$

where

$$\boldsymbol{u} \equiv \boldsymbol{v} - \boldsymbol{v}',$$

and the velocity diffusion tensor is

$$\boldsymbol{D}_a = 2 \sum_b \frac{c_{ab}}{m_a^2} \int \mathrm{d}^3v' f_b(\boldsymbol{v}') \left(\frac{\boldsymbol{I}}{u} - \frac{\boldsymbol{uu}}{u^3} \right) \equiv \sum_b \boldsymbol{D}_{ab}. \tag{3}$$

Each term in the summations over the index b represents the effect of scattering of particles of species a by particles of another species b. Like-species scattering is contained in the terms with $b = a$. Dyadic notation has been used for the second-rank tensors, with $\boldsymbol{I}$ being the unit dyadic, equivalent to the Kronecker delta in subscript notation. Also,

$$c_{ab} \equiv 2\pi e_a^2 e_b^2 \ln \Lambda. \tag{4}$$

The Coulomb logarithm is defined (Spitzer, 1962) as

$$\ln \Lambda = \ln(\lambda_{\mathrm{D}}/b_0) \tag{5}$$

where

$$\lambda_{\mathrm{D}} = \left(T/4\pi n_e e^2\right)^{1/2}$$

is the Debye shielding distance, and

$$b_0 \equiv e^2/3T$$

is a typical "distance of closest approach" for a thermal particle.

An equivalent form of the Fokker–Planck equation is obtained by using

$$\frac{\partial}{\partial \boldsymbol{v}} \cdot \left(\frac{\boldsymbol{I}}{u} - \frac{\boldsymbol{uu}}{u^3} \right) = \frac{-2\boldsymbol{u}}{u^3} = -\frac{\partial}{\partial \boldsymbol{v}'} \cdot \left(\frac{\boldsymbol{I}}{u} - \frac{\boldsymbol{uu}}{u^3} \right)$$

and integrating by parts in the expression for $\boldsymbol{A}_a$. The result,

$$\frac{\partial f_a}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}} \cdot \sum_b \frac{c_{ab}}{m_a} \int \mathrm{d}^3v' \left(\frac{\boldsymbol{I}}{u} - \frac{\boldsymbol{uu}}{u^3} \right) \cdot \left[\frac{1}{m_a} \frac{\partial f_a}{\partial \boldsymbol{v}} f_b(\boldsymbol{v}') - \frac{1}{m_b} \frac{\partial f_b}{\partial \boldsymbol{v}'} f_a(\boldsymbol{v}) \right], \tag{6}$$

is known as the Landau equation (Landau, 1936).

This result was obtained by retaining, in the Boltzmann collision integral, only the small-angle scattering events, and introducing a cut-off at the Debye shielding distance in the integral over impact parameters, which would otherwise be logarithmically divergent for the Coulomb scattering law. Although the Boltzmann equation was originally intended to describe a gas of particles interacting through binary collisions, it may be applied to a plasma, in which many charged particles interact simultaneously, if the effects of three-particle correlations are negligible. Many simultaneous small-angle scattering events then have the same averaged effect on a particle as a sequence of independent binary collisions. The two-particle correlation effectively cuts off the Coulomb potential at a distance of about one Debye length.

A somewhat more fundamental derivation of the Fokker–Planck equation can be given, using the analogy between charged-particle diffusion in velocity space and Brownian motion. The form of the Fokker–Planck equation follows from the assumption that velocity diffusion is a Markov process (Chandrasekhar, 1943a, b). Specific expressions for the Fokker–Planck coefficients $\boldsymbol{A}_{ab}$, $\boldsymbol{D}_{ab}$ are then obtained by again using the Coulomb binary scattering law. These coefficients can be written (Rosenbluth et al. 1957; Trubnikov, 1958, 1965) in the form

$$\boldsymbol{A}_{ab} = 2 \frac{c_{ab}}{m_a^2} \left(1 + \frac{m_a}{m_b}\right) \frac{\partial h_b}{\partial \boldsymbol{v}} \tag{7}$$

$$\boldsymbol{D}_{ab} = 2 \frac{c_{ab}}{m_a^2} \frac{\partial^2 g_b}{\partial \boldsymbol{v}\, \partial \boldsymbol{v}} \tag{8}$$

where

$$g_b(\boldsymbol{v}) = \int \mathrm{d}^3v' f_b(\boldsymbol{v}') |\boldsymbol{v} - \boldsymbol{v}'| \tag{9}$$

and

$$h_b(\boldsymbol{v}) = \int \mathrm{d}^3v' \frac{f_b(\boldsymbol{v}')}{|\boldsymbol{v}-\boldsymbol{v}'|}. \tag{10}$$

By using

$$\frac{\partial}{\partial \boldsymbol{v}}\frac{1}{u} = -\frac{\boldsymbol{u}}{u^3}$$

and

$$\frac{\partial^2 u}{\partial \boldsymbol{v}\partial \boldsymbol{v}} = \frac{\boldsymbol{I}}{u} - \frac{\boldsymbol{uu}}{u^3}$$

where

$$\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}',$$

(7) and (8) are easily seen to be equivalent to (2) and (3).

The functions g_b, h_b are called *Rosenbluth potentials*, because of an electrostatic analogy. By using

$$\nabla_v^2 u = 2/u$$

and

$$\nabla_v^2(1/u) = -4\pi\delta(\boldsymbol{u})$$

where ∇_v^2 is the Laplacian operator in velocity space, and $\delta(\boldsymbol{u})$ is a delta function,

$$\nabla_v^2 g_b = 2h_b \tag{11}$$

$$\nabla_v^2 h_b = -4\pi f_b. \tag{12}$$

Thus, h_b is analogous to the potential in real space due to a charge density f_b, g_v is the potential which results from considering $-h_b/2\pi$ as the charge density. It follows from this analogy that if f_b is spherically symmetric then g_b and h_b are also, and that $h_b(v)$ and $g_b(v)$ are affected only by $f_b(v')$ with $v' \leqslant v$. Thus, the dynamical friction vector $\boldsymbol{A}_a(v)$ and velocity diffusion tensor $\boldsymbol{D}_a(v)$ acting on a particle with a given speed v, are determined only by interactions with slower particles (Belyaev and Budker, 1956). It also follows that the dynamical friction force on a fast electron decreases as v^{-2}, by analogy with the electric field outside a spherically symmetric charge distribution.

The right-hand side of the Fokker–Planck equation can be written as follows:

$$\frac{\partial f_a}{\partial t} = \sum_b C_{ab}[f_a, f_b] \tag{13}$$

where the Fokker–Planck collision terms can be written, using (11), as

$$C_{ab}[f_a, f_b] = -\Gamma_a Z_b^2 \frac{\partial}{\partial \boldsymbol{v}}\cdot\left(\frac{m_a}{m_b}\frac{\partial h_b}{\partial \boldsymbol{v}} f_a - \frac{1}{2}\frac{\partial^2 g_b}{\partial \boldsymbol{v}\partial \boldsymbol{v}}\cdot\frac{\partial f_a}{\partial \boldsymbol{v}}\right) \tag{14}$$

where

$$\Gamma_a = \frac{4\pi z_a^2 e^4 \ln\Lambda}{m_a^2}. \tag{15}$$

At this point it may be well to mention that in the derivations referred to above, it was necessary to assume that $\ln\Lambda \gg 1$. Since Λ is proportional to the number of particles in a Debye sphere (i.e. a sphere whose radius is a Debye length), this number must be very large. When it is, the average particle kinetic energy is much larger than the average potential energy of two charged particles. The many small-angle scattering events then have a greater cumulative effect than the few large-angle scattering events. For most densities and temperatures of interest $\ln\Lambda$ is in the range 15–20, so this assumption is justified.

A still more fundamental derivation of the Fokker–Planck equation starts by taking into account that the Coulomb interaction between two particles is modified by the rest of the plasma in a more complicated way than by static Debye shielding. Thus, a moving electron has a distorted shielding cloud, and a rapidly moving electron radiates Langmuir waves by Cerenkov emission. The Balescu–Lenard equation (Balescu, 1960; Lenard, 1960; Hubbard, 1961) which includes these effects, differs from the Landau equation, (6), by the replacement of

$$\ln\Lambda\left(\frac{\boldsymbol{I}}{|\boldsymbol{v}-\boldsymbol{v}'|} - \frac{(\boldsymbol{v}-\boldsymbol{v}')(\boldsymbol{v}-\boldsymbol{v}')}{|\boldsymbol{v}-\boldsymbol{v}'|^3}\right) \tag{16}$$

by

$$\frac{1}{\pi}\int \mathrm{d}^3k \frac{\boldsymbol{kk}\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')}{k^4|\varepsilon(\boldsymbol{k}\cdot\boldsymbol{v}, \boldsymbol{k})|^2}. \tag{17}$$

Here $\varepsilon(\omega, \boldsymbol{k})$ is the longitudinal dielectric function, whose zeros determine the dispersion relations for electrostatic waves. In the integral over wavenumbers, which corresponds roughly to the Boltzmann integral over impact parameters, there is no need for an artificial cut-off at small values of k, since the integral converges (because $\varepsilon \sim k^{-2}$ for $k \to 0$). For particle velocities which are not much larger than the root-mean-square value, we may use the static dielectric constant

$$\varepsilon(0, \boldsymbol{k}) = 1 + 1/k^2\lambda_D^2$$

and then (17) reduces to (16). Thus, the Landau equation is recovered, with the Debye length appearing naturally in the Coulomb logarithm, without being introduced artificially as a cut-off.

For higher-velocity particles, wave-particle effects, which are not included in the Landau equation, may be important. However, these may not be described correctly by the Balescu–Lenard equation either. This is because Langmuir waves with high phase velocities are very weakly damped, and take on a "life of their own", especially in a non-thermal equilibrium plasma (Rogister and Oberman, 1968). Thus, a separate equation governing the time dependence of the wave amplitudes is necessary. Since the equations which result are too complicated to be useful in most transport applications, we simply omit this portion of the wave spectrum.

The Landau form of the Fokker–Planck equation is also commonly used in applications in which there is an external magnetic field present. A generalization of the Balescu–Lenard equation which includes the magnetic field was derived by Rostoker and Rosenbluth (1960) but it is much more complicated than the Landau equation. Montgomery et al. (1974) have concluded that the only change in the Landau collision term needed, when the mean electron gyroradius is much smaller than the Debye length, is the replacement, in the Coulomb logarithm, of the latter by the former. In using the Landau equation, we do not include the transport caused by thermally excited convective cells (Taylor and McNamara, 1971; Okuda and Dawson, 1973) although it may be more important than collisional diffusion in some situations.

Properties of the Fokker–Planck collision terms

The Fokker–Planck equation has several important general properties which are important in transport theory applications, which we now demonstrate.

Positivity of the distribution function. That f_a cannot become negative, if it is non-negative initially, can be seen as follows. Suppose f_a went to zero at a single point $\boldsymbol{v} = \boldsymbol{v}_0$; then $\partial f_a/\partial \boldsymbol{v} = 0$ there also, and

$$f_a(\boldsymbol{v}) \simeq \tfrac{1}{2}(\boldsymbol{v}-\boldsymbol{v}_0)\cdot(\partial^2 f/\partial\boldsymbol{v}\partial\boldsymbol{v})_0\cdot(\boldsymbol{v}-\boldsymbol{v}_0) > 0$$

for small $|\boldsymbol{v}-\boldsymbol{v}_0| \neq 0$. The tensor $(\partial^2 f/\partial\boldsymbol{v}\partial\boldsymbol{v})_0$ must therefore be positive-definite. Equation (6) at $\boldsymbol{v} = \boldsymbol{v}_0$ then becomes

$$\frac{\partial f_a}{\partial t} = \sum_b \frac{c_{ab}}{m_a^2} \int \mathrm{d}^3v' f_b(\boldsymbol{v}') \left(\frac{\boldsymbol{I}}{u} - \frac{\boldsymbol{uu}}{u^3} \right) : \left(\frac{\partial^2 f_a}{\partial\boldsymbol{v}\partial\boldsymbol{v}} \right)_0$$

where $\boldsymbol{u} = \boldsymbol{v}_0 - \boldsymbol{v}'$. By defining $\hat{e}_1 \equiv \boldsymbol{u}/u$, with $\hat{e}_2, \hat{e}_3$ any two other mutually perpendicular unit vectors, we have

$$\frac{\boldsymbol{I}}{u} - \frac{\boldsymbol{uu}}{u^3} = \frac{\hat{e}_2\hat{e}_2 + \hat{e}_3\hat{e}_3}{u}$$

so that

$$\frac{\partial f_a}{\partial t} = \sum_b \frac{c_{ab}}{m_a^2} \int \mathrm{d}^3v' \frac{f_b(\boldsymbol{v}')}{u} \left[\hat{e}_2\cdot\left(\frac{\partial^2 f_a}{\partial\boldsymbol{v}\partial\boldsymbol{v}}\right)_0\cdot\hat{e}_e + \hat{e}_3\cdot\left(\frac{\partial^2 f_a}{\partial\boldsymbol{v}\partial\boldsymbol{v}}\right)_0\cdot\hat{e}_3 \right] > 0.$$

Thus, f_a becomes positive at $\boldsymbol{v} = \boldsymbol{v}_0$: collisions tend to "fill in" any holes in velocity space.

Particle conservation. By writing (6) in the form

$$\frac{\partial f_a}{\partial t} = \frac{\partial}{\partial\boldsymbol{v}}\cdot\boldsymbol{J}_a \tag{18}$$

and integrating over velocity space, the divergence theorem can be used to show that

$$\frac{\partial}{\partial t}\int \mathrm{d}^3v f_a = \int \mathrm{d}^3v \frac{\partial}{\partial\boldsymbol{v}}\cdot\boldsymbol{J}_a = 0 \tag{19}$$

where it has been assumed that $f_a \to 0$ sufficiently fast, as $v \to \infty$, that the surface term at $v = \infty$ is zero.

The remaining properties are most conveniently derived using the following alternative form of (13):

$$\frac{\partial f_a}{\partial t} = \sum_b C_{ab} \tag{20}$$

where

$$C_{ab} = \frac{c_{ab}}{m_a}\frac{\partial}{\partial\boldsymbol{v}}\cdot\int \mathrm{d}^3v' f_a(\boldsymbol{v}) f_b(\boldsymbol{v}') \left(\frac{\boldsymbol{I}}{u} - \frac{\boldsymbol{uu}}{u^3} \right)\cdot\boldsymbol{\chi}_{ab}(\boldsymbol{v},\boldsymbol{v}') \tag{21}$$

where $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}'$, and

$$\boldsymbol{\chi}_{ab}(\boldsymbol{v},\boldsymbol{v}') = \frac{1}{m_a}\frac{\partial}{\partial\boldsymbol{v}}\ln f_a - \frac{1}{m_b}\frac{\partial}{\partial\boldsymbol{v}'}\ln f_b. \tag{22}$$

Momentum conservation. Multiplying (20) by $m_a\boldsymbol{v}$ and integrating over velocity yields:

$$\frac{\partial}{\partial t}\int \mathrm{d}^3v\, m_a\boldsymbol{v} f_a = \sum_b \boldsymbol{F}_{ab} \tag{23}$$

where

$$\boldsymbol{F}_{ab} \equiv \int \mathrm{d}^3v\, m_a\boldsymbol{v} C_{ab} = -c_{ab}\int \mathrm{d}^3v \int \mathrm{d}^3v' f_a(\boldsymbol{v}) f_b(\boldsymbol{v}') \left(\frac{\boldsymbol{I}}{u} - \frac{\boldsymbol{uu}}{u^3} \right)\cdot\boldsymbol{\chi}_{ab}(\boldsymbol{v},\boldsymbol{v}') \tag{24}$$

is the momentum transfer rate, or friction force. Because $c_{ab} = c_{ba}$, and because the integrand simply changes sign upon making the replacements $a \leftrightarrow b$, $\boldsymbol{v} \leftrightarrow \boldsymbol{v}'$, we have

$$\boldsymbol{F}_{ab} = -\boldsymbol{F}_{ba}. \tag{25}$$

That is, the collisional momentum transfer from species b to species a is equal in magnitude and opposite in direction to that from a to b, which reaffirms Newton's third law. In particular, with $b = a$,

$$\boldsymbol{F}_{aa} = 0. \tag{26}$$

That is, the total momentum of a given species of particles is conserved by like-species collisions. Summing (23) over all species a, and using (25) yields:

$$\frac{\partial}{\partial t}\sum_a \int \mathrm{d}^3v\, m_a\boldsymbol{v} f_a = \sum_a\sum_b \boldsymbol{F}_{ab} = 0. \tag{27}$$

That is, the total momentum, including all species, is conserved by collisions.

Energy conservation. Multiplying (20) by $\frac{1}{2}m_a(\boldsymbol{v}-\boldsymbol{u}_a)^2$ and integrating over velocity yields:

$$\frac{\partial}{\partial t}\int d^3v\,\tfrac{1}{2}m_a(\boldsymbol{v}-\boldsymbol{u}_a)^2 f_a=\sum_b Q_{ab}=\sum_b\left(\int d^3v\frac{m_a v^2}{2}C_{ab}-\boldsymbol{u}_a\cdot\boldsymbol{F}_{ab}\right) \tag{28}$$

where

$$Q_{ab}+\boldsymbol{u}_a\cdot\boldsymbol{F}_{ab}=-c_{ab}\int d^3v\int d^3v' f_a(\boldsymbol{v})f_b(\boldsymbol{v}')\boldsymbol{v}\cdot\left(\frac{\boldsymbol{I}}{u}-\frac{\boldsymbol{uu}}{u^3}\right):\chi_{ab}(\boldsymbol{v},\boldsymbol{v}') \tag{29}$$

is the energy transfer rate. Since $c_{ab}=+c_{ba}$, upon making the replacements $a\leftrightarrow b$, $\boldsymbol{v}\leftrightarrow\boldsymbol{v}'$,

$$Q_{ba}+\boldsymbol{u}_b\cdot\boldsymbol{F}_{ba}=c_{ab}\int d^3v\int d^3v' f_a(\boldsymbol{v})f_b(\boldsymbol{v}')\boldsymbol{v}'\cdot\left(\frac{\boldsymbol{I}}{u}-\frac{\boldsymbol{uu}}{u^3}\right)\cdot\chi_{ab}(\boldsymbol{v},\boldsymbol{v}'). \tag{30}$$

Adding (29) and (30), using

$$(\boldsymbol{v}'-\boldsymbol{v})\cdot\left(\frac{\boldsymbol{I}}{u}-\frac{\boldsymbol{uu}}{u^3}\right)=0$$

yields:

$$Q_{ab}+Q_{ba}=(\boldsymbol{u}_b-\boldsymbol{u}_a)\cdot\boldsymbol{F}_{ab} \tag{31}$$

which expresses the conservation of energy in collisions between species a and b. In particular, if $b=a$, then

$$Q_{aa}=0. \tag{32}$$

That is, the total energy of a given species of particles is conserved by like-species collisions. Summing (28) over all species a, using (31), yields:

$$\frac{\partial}{\partial t}\sum_a\int d^3v\frac{m_a v^2}{2}f_a=\sum_a\sum_b(Q_{ab}+\boldsymbol{u}_a\cdot\boldsymbol{F}_{ab})=0. \tag{33}$$

That is, the total energy, including all species, is conserved by collisions.

The H-theorem. That the only time-independent distribution functions are Maxwellians can be proved as follows. The entropy density is defined as

$$s=-\sum_a\int d^3v f_a(\ln f_a+k_a) \tag{34}$$

where k_a is a constant determined by quantum mechanical considerations (Landau and Lifshitz, 1958). Differentiating (34), using particle conservation, yields:

$$\frac{\partial s}{\partial t}=-\sum_a\int d^3v\ln f_a\frac{\partial f_a}{\partial t}. \tag{35}$$

Using (20) for $\partial f_a/\partial t$ and integrating by parts yields:

$$\frac{\partial s}{\partial t}=\sum_a\sum_b\frac{c_{ab}}{m_a}\int d^3v\int d^3v' f_a(\boldsymbol{v})f_b(\boldsymbol{v}')\left(\frac{\partial}{\partial\boldsymbol{v}}\ln f_a\right)\cdot\left(\frac{\boldsymbol{I}}{u}-\frac{\boldsymbol{uu}}{u^3}\right)\cdot\chi_{ab}(\boldsymbol{v},\boldsymbol{v}'). \tag{36}$$

The right-hand side of (36) can be written in an alternative form by making the replacements $a\leftrightarrow b$, $\boldsymbol{v}\leftrightarrow\boldsymbol{v}'$; adding these two forms yields:

$$\frac{\partial s}{\partial t}=\frac{1}{2}\sum_a\sum_b c_{ab}\int d^3v\int d^3v' f_a(\boldsymbol{v})f_b(\boldsymbol{v}')\chi_{ab}\cdot\left(\frac{\boldsymbol{I}}{u}-\frac{\boldsymbol{uu}}{u^3}\right)\cdot\chi_{ab}\geqslant 0, \tag{37}$$

where the non-negative property follows from

$$\chi\cdot\left(\frac{\boldsymbol{I}}{u}-\frac{\boldsymbol{uu}}{u^3}\right)\cdot\chi=\frac{\chi^2}{u}(1-\cos^2\alpha)\geqslant 0$$

where $\cos\alpha=\chi\cdot\boldsymbol{u}/(|\chi|u)$. Equality in (37), which is equivalent to the condition for a time-independent solution of (20), requires that

$$\chi_{ab}(\boldsymbol{v},\boldsymbol{v}')\equiv\frac{1}{m_a}\frac{\partial}{\partial\boldsymbol{v}}\ln f_a-\frac{1}{m_b}\frac{\partial}{\partial\boldsymbol{v}'}\ln f_b=G_{ab}(\boldsymbol{v},\boldsymbol{v}')(\boldsymbol{v}-\boldsymbol{v}')$$

for some scalar function $G_{ab}(\boldsymbol{v},\boldsymbol{v}')$. Taking the curl in velocity space yields:

$$\frac{\partial}{\partial\boldsymbol{v}}\times\chi_{ab}=0=\frac{\partial G_{ab}}{\partial\boldsymbol{v}}\times(\boldsymbol{v}-\boldsymbol{v}')$$

which implies that

$$G_{ab}=G_{ab}(|\boldsymbol{v}-\boldsymbol{v}'|).$$

By setting $\boldsymbol{v}'=0$ and then $\boldsymbol{v}=0$, it is not hard to show (Montgomery and Tidman, 1964) that

$$\frac{1}{m_a}\left(\frac{\partial}{\partial\boldsymbol{v}}\ln f_a\right)_{\boldsymbol{v}=0}=\frac{1}{m_b}\left(\frac{\partial}{\partial\boldsymbol{v}'}\ln f_b\right)_{\boldsymbol{v}'=0}\equiv\boldsymbol{\beta}$$

and

$$G_{ab}(|\boldsymbol{v}-\boldsymbol{v}'|)=\gamma=\text{constant}.$$

It follows that

$$\frac{1}{m_a}\frac{\partial}{\partial\boldsymbol{v}}\ln f_a=\boldsymbol{\beta}+\gamma\boldsymbol{v} \tag{38}$$

which, upon integrating, gives

$$\ln f_a=m_a\left(\alpha_a+\boldsymbol{\beta}\cdot\boldsymbol{v}+\gamma v^2/2\right). \tag{39}$$

That is, the distribution functions must be Maxwellians with a common mean velocity and temperature:

$$f_a=n_a(m_a/2\pi T)^{3/2}\exp\left[-m_a(\boldsymbol{v}-\boldsymbol{u})^2/2T\right]. \tag{40}$$

Macroscopic relaxation times

When two different particle species are not in thermal equilibrium with each other, collisions tend to make their distribution functions relax towards thermal equilibrium. An estimate of the relaxation times involved is obtained by calculating the collisional rate of change of moments of these distributions, approximating them by Maxwellians.

Collisional momentum transfer rate. The momentum transfer rate, defined by (24), can be written in terms of the dynamical friction vector:

$$\boldsymbol{F}_{ab} = m_a \int d^3v\, \boldsymbol{A}_{ab} f_a(\boldsymbol{v}). \tag{41}$$

Using the definition of $\boldsymbol{A}_{ab}$, (7), and integrating by parts yields:

$$\boldsymbol{F}_{ab} = -z_b^2 \Gamma_a m_a (1 + m_a/m_b) \int d^3v\, h_b(\boldsymbol{v}) \frac{\partial f_a}{\partial \boldsymbol{v}}. \tag{42}$$

By substituting a Maxwellian with mean velocity $\boldsymbol{u}_b$,

$$f_b(\boldsymbol{v}) = (n_b/\pi^{3/2} v_b^3) \exp[-(\boldsymbol{v} - \boldsymbol{u}_b)^2 / v_b^2] \tag{43}$$

into the definition of h_b, (10), the integral can be carried out to give

$$h_b(\boldsymbol{v}) = \frac{n_b}{v_b} \frac{\Phi(|\boldsymbol{y}|)}{|\boldsymbol{y}|} \tag{44}$$

where $\boldsymbol{y} = (\boldsymbol{v} - \boldsymbol{u}_b)/v_b$, and $\Phi(x)$ is the error function,

$$\Phi(x) \equiv \frac{2}{\pi^{1/2}} \int_0^x dy\, e^{-y^2}. \tag{45}$$

f_a is taken to be a Maxwellian with mean velocity $\boldsymbol{u}_a$, and expressed in terms of the variable $\boldsymbol{y}$, assuming that

$$|\boldsymbol{u}_b - \boldsymbol{u}_a| \ll v_a$$

to expand it to first order in $\boldsymbol{u}_b - \boldsymbol{u}_a$. Then by using $\boldsymbol{y}$ as the integration variable in (42), the integral is evaluated using standard integral formulas to obtain

$$\boldsymbol{F}_{ab} = m_a n_a (\boldsymbol{u}_b - \boldsymbol{u}_a)/\tau_{ab} \tag{46}$$

where the momentum-transfer time is defined by

$$\tau_{ab}^{-1} = \frac{4}{3\pi^{1/2}} \frac{n_b z_b^2 \Gamma_a (1 + m_a/m_b)}{(v_a^2 + v_b^2)^{3/2}} \tag{47}$$

where $z_a^2 \equiv e_a^2/e^2$, $v_a^2 = 2T_a/m_a$ and Γ_a is defined by (15). Note that, since

$$m_a n_a/\tau_{ab} = m_b n_b/\tau_{ba}, \tag{48}$$

$$\boldsymbol{F}_{ab} = -\boldsymbol{F}_{ba}, \tag{49}$$

consistent with momentum conservation.

Thus, τ_{ab} gives the order of magnitude of the time for enough momentum to be transferred between two particle species to eliminate the difference in their mean velocities. It should be emphasized that (46) gives only a rough approximation, since quantitative results for the momentum transfer rate can generally not be obtained without considering the distortion of the distribution functions from Maxwellians.

Collisional energy exchange rate. The energy exchange rate, defined by (28) can be written in terms of the dynamical friction vector and the velocity diffusion tensor:

$$Q_{ab} = m_a \int d^3v [(\boldsymbol{v} - \boldsymbol{u}_a)\cdot \boldsymbol{A}_{ab} + \tfrac{1}{2} \mathrm{Tr} \boldsymbol{D}_{ab}] f_a \tag{50}$$

where "Tr" denotes the trace. By using the definitions (7), (8) and (11),

$$Q_{ab} = z_b^2 \Gamma_a m_a \int d^3v \left[\left(1 + \frac{m_a}{m_b}\right) (\boldsymbol{v} - \boldsymbol{u}_a)\cdot \frac{\partial h_b}{\partial \boldsymbol{v}} + h_b \right] f_a. \tag{51}$$

Again taking f_a and f_b to be Maxwellians (but with zero mean velocities this time) (44) may be used for the Rosenbluth potential h_b; after carrying out the integrations,

$$Q_{ab} = \frac{3 n_a m_a (T_b - T_a)}{\tau_{ab} (m_b + m_a)} \tag{52}$$

where τ_{ab} is given by (47), which was first derived by Spitzer (1940). Note that, from (48),

$$Q_{ab} = -Q_{ba}, \tag{53}$$

consistent with energy conservation.

The approach to equilibrium in a simple plasma. In a simple plasma, i.e. one with one ion species, the rates at which electrons and ions equilibrate separately and with each other may be compared. By regarding a and b in the above formulas as labeling two different components of the distribution of a given species, e.g. electrons, it can be seen that these parts exchange both energy and momentum on a time scale τ_{aa} given by

$$\tau_{aa}^{-1} = \frac{4}{3} \pi^{1/2} \frac{n_a z_a^4 e^4 \ln \Lambda}{m_a^{1/2} T_a^{3/2}}. \tag{54}$$

Thus, the ions come to equilibrium among themselves, with their distribution function approaching a Maxwellian, at a rate τ_{ii}^{-1}, which is considerably slower than the rate τ_{ee}^{-1} at which the electrons Maxwellianize, since

$$\tau_{ii}^{-1} \sim (m_e/m_i)^{1/2} \tau_{ee}^{-1} \tag{55}$$

(assuming $T_e \sim T_i$). The exchange of energy, and consequently the temperature relaxation between electrons and ions, occurs at a still slower rate, of the order of $(m_e/m_i)\tau_{ee}^{-1}$. The rate of transfer of momentum, and the decay of the difference in mean velocities, $\boldsymbol{u}_e - \boldsymbol{u}_i$, would occur at a rate

$$\tau_{ei}^{-1} = \frac{4}{3} (2\pi)^{1/2} \frac{n_i z_i^2 e^4 \ln \Lambda}{m_e^{1/2} T_e^{3/2}} \tag{56}$$

(which is of the same order of magnitude as τ_{ee}^{-1}) except for electromagnetic induction, which tends to prevent the current density from changing that rapidly.

Scattering of a test particle in a plasma

The velocity diffusion process described by the Fokker–Planck equation can be most easily understood by considering a test particle, that is a particle whose velocity is assumed to be known at some initial time. The distribution function for this test particle will be normalized so that its spatial number density is

$$\int \mathrm{d}^3 v\, f_t = n_t . \tag{57}$$

The test particle constitutes a particle "species" which is labeled by "t". We assume that f_t is spatially homogeneous, so that spatial gradients and electric fields do not enter the problem. The equation to be solved is then

$$\frac{\partial f_t}{\partial t} = -\frac{\partial}{\partial \boldsymbol{v}} \cdot \left[\boldsymbol{A}_t f_t - \frac{1}{2} \frac{\partial}{\partial \boldsymbol{v}} \cdot (\boldsymbol{D}_t f_t) \right] \tag{58}$$

The sum over b in the coefficients $\boldsymbol{A}_t$ and $\boldsymbol{D}_t$, as given by (2) and (3), will not include the test particle species, assuming that the spatial density n_t is small compared with the densities of the plasma species. Also, the distribution functions for the other species will be assumed to be given functions. Thus, (58) becomes a linear equation for f_t.

Suppose that the test particle is known to have velocity $\boldsymbol{v}_0$ at time $t = 0$, but its spatial position is completely unknown:

$$f_t = n_t \delta(\boldsymbol{v} - \boldsymbol{v}_0) \qquad \text{at } t = 0. \tag{59}$$

For short enough times, the test particle's velocity should remain well defined:

$$f_t(\boldsymbol{v}, t) \simeq n_t \delta(\boldsymbol{v} - \boldsymbol{u}(t)) \tag{60}$$

where $\boldsymbol{u}(t)$ is a function to be determined, which is such that $\boldsymbol{u}(0) = \boldsymbol{v}_0$. Inserting this expression into (58), multiplying by $\boldsymbol{v}$ and integrating over velocity space yields:

$$\mathrm{d}\boldsymbol{u}/\mathrm{d}t = \boldsymbol{A}_t(\boldsymbol{u}) \tag{61}$$

which is a differential equation for the expectation value of the test particle's velocity.

The test particle's velocity does not remain exactly well defined for $t \neq 0$, but becomes uncertain, because Coulomb scattering causes it to walk randomly away from its initial value. A better approximation to the solution for f_t can be obtained as follows. Let

$$f_t(\boldsymbol{v}, t) = n_t F_t(\boldsymbol{w}, t) \tag{62}$$

where

$$\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}(t) \tag{63}$$

and $\boldsymbol{u}(t)$ changes in time according to (61). Since F_t should remain sharply peaked about $\boldsymbol{w} = 0$ for short times, like the delta-function approximation of (60), we may

replace $\boldsymbol{v}$ in $\boldsymbol{A}_t(\boldsymbol{v})$ and $\boldsymbol{D}_t(\boldsymbol{v})$ by $\boldsymbol{u}(t)$. This leads to the equation

$$\frac{\partial F_t}{\partial t} = \frac{1}{2} \boldsymbol{D}_t(\boldsymbol{u}) : \frac{\partial^2 F_t}{\partial \boldsymbol{w} \partial \boldsymbol{w}} \tag{64}$$

where the dots indicate sums over indices. Using the initial condition

$$F_t(\boldsymbol{w}, 0) = \delta(\boldsymbol{w}),$$

the solution (Chandrasekhar, 1943a, b) can be obtained by Fourier transforming in the variable $\boldsymbol{w}$. It is a Gaussian:

$$F_t(\boldsymbol{w}, t) = \frac{\exp\left[-\frac{1}{2} \boldsymbol{w} \cdot \boldsymbol{M}^{-1}(t) \cdot \boldsymbol{w} \right]}{(2\pi)^{3/2} [\det \boldsymbol{M}(t)]^{1/2}} \tag{65}$$

where det $\boldsymbol{M}$ is the determinant of the matrix

$$\boldsymbol{M}(t) = \int_0^t \mathrm{d}\tau\, \boldsymbol{D}_t(\boldsymbol{u}(\tau)) \tag{66}$$

and $\boldsymbol{M}^{-1}$ is its inverse.

The uncertainty in the test-particle's velocity after time t can now be obtained from moments of F_t. Let $\hat{\boldsymbol{e}}_1$ be a unit vector parallel to the initial velocity $\boldsymbol{v}_0$, and let $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$ be two other mutually perpendicular unit vectors. Then the uncertainties in velocity in directions perpendicular or parallel to the initial velocity, defined by

$$\langle (\Delta v_\perp)^2 \rangle \equiv \int \mathrm{d}^3 v \left[(\boldsymbol{v} \cdot \hat{\boldsymbol{e}}_2)^2 + (\boldsymbol{v} \cdot \hat{\boldsymbol{e}}_3)^2 \right] F_t(\boldsymbol{v} - \boldsymbol{u}, t) \tag{67}$$

$$\langle (\Delta v_\parallel)^2 \rangle \equiv \int \mathrm{d}^3 v \left[(\boldsymbol{v} - \boldsymbol{u}) \cdot \hat{\boldsymbol{e}}_1 \right]^2 F_t(\boldsymbol{v} - \boldsymbol{u}, t) \tag{68}$$

are determined (for short times) by

$$(\mathrm{d}/\mathrm{d}t) \langle (\Delta v_\perp)^2 \rangle = \hat{\boldsymbol{e}}_2 \cdot \boldsymbol{D}_t \cdot \hat{\boldsymbol{e}}_2 + \hat{\boldsymbol{e}}_3 \cdot \boldsymbol{D}_t \cdot \hat{\boldsymbol{e}}_3 \tag{69}$$

$$(\mathrm{d}/\mathrm{d}t) \langle (\Delta v_\parallel)^2 \rangle = \hat{\boldsymbol{e}}_1 \cdot \boldsymbol{D}_t \cdot \hat{\boldsymbol{e}}_1 . \tag{70}$$

In the important special case of isotropic field particle distributions, the friction and diffusion coefficients have the form

$$\boldsymbol{A}_t(\boldsymbol{v}) = -\nu_s^t(v) \boldsymbol{v} \tag{71}$$

where

$$\nu_s^t(v) = -\Gamma_t \sum_b z_b^2 \left(1 + \frac{m_t}{m_b} \right) \frac{1}{v} \frac{\mathrm{d}h_b}{\mathrm{d}v} \tag{72}$$

is the slowing-down rate, and

$$\boldsymbol{D}_t(\boldsymbol{v}) = D_\perp^t(v) \left(\boldsymbol{I} - \frac{\boldsymbol{v}\boldsymbol{v}}{v^2} \right) + D_\parallel^t(v) \frac{\boldsymbol{v}\boldsymbol{v}}{v^2} \tag{73}$$

where

$$D_\perp^t(v) = (\Gamma_t/v)\sum_b z_b^2 \frac{dg_b}{dv} \tag{74}$$

$$D_\parallel^t(v) = \Gamma_t \sum_b z_b^2 \frac{d^2 g_b}{dv^2} \tag{75}$$

are the perpendicular and parallel diffusion coefficients, with

$$\Gamma_t \equiv 4\pi z_t^2 e^4 (\ln \Lambda)/m_t^2 \tag{76}$$

where $g_b(v)$ and $h_b(v)$ are the Rosenbluth potentials, defined by (9) and (10). In this case, the rate of change of the short-time uncertainties in the test particle's velocity are, from (69) and (70),

$$(d/dt)\langle(\Delta v_\perp)^2\rangle = 2D_\perp^t(u) \tag{77}$$

$$(d/dt)\langle(\Delta v_\parallel)^2\rangle = D_\parallel^t(u). \tag{78}$$

The rate of change of the expectation value of a test particle's kinetic energy is related to the slowing-down rate ν_s and the diffusion coefficients $D_\perp$ and $D_\parallel$:

$$(d/dt)\left\langle \tfrac{1}{2}m_t(\boldsymbol{u}+\Delta\boldsymbol{v})^2\right\rangle = -mu^2\nu_s^t + m_t\left(D_\perp^t + \tfrac{1}{2}D_\parallel^t\right) \equiv -\nu_E^t \tfrac{1}{2}m_t u^2 \tag{79}$$

which defines the energy-loss rate ν_E. This rate is not necessarily positive: a fast test particle tends to lose energy, but a slow one tends to gain energy.

The test-particle form of the Fokker–Planck equation can be written more simply when the field particles' distributions are isotropic. By using (71) and (73), the equation can be written in spherical velocity space coordinates as

$$\frac{\partial f_t}{\partial t} = \frac{D_\perp^t(v)}{v^2}\mathfrak{L}f_t + \frac{1}{v^2}\frac{\partial}{\partial v}\left[v^2\left(F^t f_t + \tfrac{1}{2}D_\parallel^t \frac{\partial f_t}{\partial v}\right)\right] \tag{80}$$

where

$$F^t \equiv \nu_s v + \frac{D_\parallel^t - D_\perp^t}{v} + \frac{1}{2}\frac{dD_\parallel^t}{dv} \tag{81}$$

and the operator $\mathfrak{L}$ is defined by

$$2\mathfrak{L}f \equiv \frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial f}{\partial \mu} + \frac{1}{(1-\mu^2)}\frac{\partial^2 f}{\partial \zeta^2} \tag{82}$$

($2\mathfrak{L}$ is the angular part of the Laplacian operator), where v, ϑ, ζ are the spherical coordinates, and $\mu = \cos\vartheta$.

Applications of the test-particle equation

Maxwellian field particle distributions. In order to specifically evaluate the functions $\nu_s(v)$, $D_\perp(v)$ and $D_\parallel(v)$, the further specialization is made of assuming that the plasma electrons and ions have Maxwellian velocity distributions:

$$f_b(v) = \frac{n_b}{\pi^{3/2}v_b^3}\exp(-v^2/v_b^2) \tag{83}$$

where

$$v_b \equiv (2T_b/m_b)^{1/2}. \tag{84}$$

Then the Rosenbluth potentials $h_b(v)$, $g_b(v)$ can be evaluated as

$$h_b(v) = n_b \Phi(v/v_b)/v \tag{85}$$

and

$$g_b(v) = n_b v_b \phi(v/v_b) \tag{86}$$

where

$$\phi(x) = (x + 1/2x)\Phi(x) + \pi^{-1/2}e^{-x^2} \tag{87}$$

and where $\Phi(x)$ is the error function, (45). Thus, from (72),

$$\nu_s^t(v) = \frac{2\Gamma_t}{v}\sum_b \frac{n_b z_b^2}{v_b^2}\left(1+\frac{m_t}{m_b}\right)G\left(\frac{v}{v_b}\right) \tag{88}$$

and from (74) and (75)

$$D_\perp^t(v) = \frac{\Gamma_t}{v}\sum_b n_b z_b^2\left[\Phi\left(\frac{v}{v_b}\right) - G\left(\frac{v}{v_b}\right)\right] \tag{89}$$

$$D_\parallel^t(v) = \frac{2\Gamma_t}{v}\sum_b n_b z_b^2 G\left(\frac{v}{v_b}\right) \tag{90}$$

where

$$G(x) = \frac{\Phi(x) - x\Phi'(x)}{2x^2} \tag{91}$$

is the Chandrasekhar function (Chandrasekhar, 1943; Spitzer, 1962). The energy-loss rate, defined by (79), is given by

$$\nu_E = \frac{2\Gamma_t m_t}{v^3}\sum_b \frac{n_b z_b^2}{m_b}\left[\Phi\left(\frac{v}{v_b}\right) - \left(1+\frac{m_b}{m_t}\right)\frac{v}{v_b}\Phi'\left(\frac{v}{v_b}\right)\right]. \tag{92}$$

For a single species of field particles b, the slowing down rate given by (88) is equal to t_s^{-1}, where t_s is the slowing-down time defined by Spitzer (1962). A deflection rate, ν_D, may also be defined by

$$\nu_D^t = 2D_\perp^t/v^2 \tag{93}$$

which in this case equals t_D^{-1}, where t_D is Spitzer's deflection time.

The form of the test-particle kinetic equation is

$$\frac{\partial f_t}{\partial t} = \frac{1}{2}\nu_D^t\mathfrak{L}f_t + \Gamma_t\sum_b \frac{n_b z_b^2}{v^2}\frac{\partial}{\partial v}\left[vG\left(\frac{v}{v_b}\right)\left(\frac{\partial f_t}{\partial v} + \frac{m_t v}{T_b}f_t\right)\right]. \tag{94}$$

If all of the field particle species have the same temperature T, then the only time-independent solution of this equation is a Maxwellian with temperature T. The test particle distribution must therefore spread out in velocity space, starting from a delta function, and ultimately become isotropic, and, in fact, Maxwellian, as it reaches thermal equilibrium with the field particles.

We now give results in limiting cases specifically for a simple plasma, in which the ions have charge $z_i e$. The test particle will be assumed to have charge $z_t e$. The slowing down rate, and the perpendicular and parallel diffusion coefficients are given by the following approximate formulas, in which

$$\Gamma_t \equiv 4\pi z_t^2 e^4 (\ln \Lambda)/m_t^2. \tag{95}$$

(a) For $v \ll v_i, v_e$:

$$\nu_s^t(v) \simeq (4/3\pi^{1/2}) n_e \Gamma_t \left[(1+m_t/m_e)/v_e^3 + z_i(1+m_t/m_i)/v_i^3\right] \tag{96}$$

$$D_\perp^t(v) \simeq D_\parallel^t(v) \simeq (4/3\pi^{1/2}) n_e \Gamma_t (1/v_e + z_i/v_i). \tag{97}$$

For very small velocities, the slowing-down rate and the diffusion coefficients are independent of velocity, so the test-particle motion is the same as Brownian motion (Chandrasekhar, 1943).

(b) For $v_i \ll v \ll v_e$:

$$\nu_s^t(v) \simeq n_e \Gamma_t \left[(4/3\pi^{1/2})(1+m_t/m_e)/v_e^3 + z_i(1+m_t/m_i)/v^3\right] \tag{98}$$

$$D_\perp^t(v) \simeq n_e \Gamma_t \left[(4/3\pi^{1/2})/v_e + z_i/v\right] \tag{99}$$

$$D_\parallel^t(v) \simeq n_e \Gamma_t \left[(4/3\pi^{1/2})/v_e + z_i v_i^2/v^3\right]. \tag{100}$$

For this range of velocities, the electron collisions have the same effect as in range (a), while the ion collisions have a much different effect. Diffusion due to ion collisions is primarily perpendicular to the test particle's velocity, and both the slowing-down rate and the diffusion coefficient decrease with test-particle velocity.

(c) For $v_i, v_e \ll v$:

$$\nu_s^t(v) \simeq n_e(\Gamma_t/v^3)\left[(1+m_t/m_e) + z_i(1+m_t/m_i)\right] \tag{101}$$

$$D_\perp^t(v) \simeq n_e(\Gamma_t/v)(1+z_i) \tag{102}$$

$$D_\parallel^t(v) \simeq n_e(\Gamma_t/v^3)(v_e^2 + z_i v_i^2). \tag{103}$$

For very high velocities, diffusion is mainly perpendicular to the velocity of the test particle, with electrons and ions making roughly equal contributions. For an ion test particle the slowing-down rate is due mainly to collisions with electrons, while for an electron test particle the electron and ion collisional contributions are roughly comparable.

The macroscopic relaxation times discussed above are consistent with the evaluation of (88) and (92) for a thermal velocity. For velocities much higher than thermal, however, the values given by (101)–(103) are much smaller. Thus, the high-energy tails of the distribution functions thermalize much more slowly than the thermal portions.

Slowing down of energetic ions. Neutral beam injection (Stix, 1971) is a method which is presently being used to heat plasmas in magnetic confinement experiments. The energetic ions which result from ionization or charge exchange of the injected neutrals then slow down because of Coulomb collisions and deposit their kinetic energy in the plasma (Sivukhin, 1966). Provided that the density of the energetic ions is not too large, their distribution function satisfies the test-particle form of the Fokker–Planck equation. Assuming the plasma to be isotropic, this takes the form (80), except for a source term at the injection velocity which would also have to be included. For a plasma in which the electrons and ions have Maxwellian distributions, ν_s is given by (88) while $D_\perp$ and $D_\parallel$ are given by (89) and (90).

For these energetic ions, the most relevant range of velocities is $v_i \ll v \ll v_e$, and then (98)–(100) may be used for ν_s, $D_\perp$ and $D_\parallel$. Using the fact that $m_t \sim m_i \gg m_e$ gives for the slowing-down rate

$$\nu_s^t(v) \simeq n_e \Gamma_t \left[\left(\frac{4}{3\pi^{1/2}}\right)\frac{m_t}{m_e v_e^3} + \left(1+\frac{m_t}{m_i}\right)\frac{z_i}{v^3}\right] \tag{104}$$

and for the parallel diffusion coefficient

$$D_\parallel(v) \simeq \frac{n_e \Gamma_t}{v_e}\left(\frac{4}{3\pi^{1/2}} + \frac{z_i v_e v_i^2}{v^3}\right) \tag{105}$$

in which the first (electron) terms dominate only for $v/v_e \gtrsim (z_i m_e/m_i)^{1/3}$, and for the perpendicular diffusion coefficient,

$$D_\perp^t(v) \simeq n_e \Gamma_t z_i / v \tag{106}$$

in which only the ion collisions are important. The energy-loss rate (92) can be approximated by

$$\nu_E \simeq \frac{2 n_e \Gamma_t m_t z_i}{m_i}\left(\frac{1}{v_c^3} + \frac{1}{v^3}\right) \tag{107}$$

where

$$v_c = \left(\frac{3\pi^{1/2}}{4}\frac{m_e z_i}{m_i}\right)^{1/3} v_e. \tag{108}$$

The energy from the injected ion goes mainly into heating electrons if $v > v_c$ and otherwise it goes mainly into heating the ions.

In (80) the parallel diffusion can be neglected compared with the perpendicular diffusion, since $D_\parallel \ll D_\perp$, so the equation becomes, with the inclusion of a source term at the injection velocity,

$$\frac{\partial f_t}{\partial t} = n_e \Gamma_t z_i \left\{\frac{1}{v^3}\mathfrak{L} f_t + \frac{m_t}{m_i v^2}\frac{\partial}{\partial v}\left[\left(\frac{v^3}{v_c^3}+1\right) f_t\right]\right\} + S\delta(v-v_0)\delta(\mu-\mu_0) \tag{109}$$

where $\mathfrak{L}$ is defined by (82) and where v_0 and μ_0 are the injection speed and the cosine of the injection angle. The source has been assumed to be symmetric about the axis of the polar coordinate system. This equation can be solved analytically (Cordey and Houghton, 1973) to give the time development of the injected ion distribution function. The solutions show diffusion in angle and a spreading in energy due to slowing down of the ions at a well-defined rate. Diffusion in energy, not included in (109), would broaden the energy distribution slightly and accelerate a few ions to energies greater than the injection energy.

Electron runaway. Although the Fokker–Planck equation was derived assuming no electric field, it is commonly employed in applications where an electric field is present. In addition to the transport (current and heat flow) driven by an electric field, which will be considered later, it is of interest to investigate the behavior of the electron distribution function at high energies. Since the fraction of electrons in the high-energy tail of the distribution is assumed to be small, the test-particle form of the Fokker–Planck equation applies. Assuming that $v \gg v_e \gg v_i$ and since $m_t = m_e$, (101)–(103) give

$$\nu_s = (n_e \Gamma_e / v^3)(2 + z_i)$$

$$D_\perp = (n_e \Gamma_e / v)(1 + z_i)$$

$$D_\parallel = n_e \Gamma_e v_e^2 / v^3.$$

Thus, (80) becomes, with the inclusion of the electric field term,

$$\frac{\partial f}{\partial t} - \frac{e}{m_e} E\left(\mu \frac{\partial f}{\partial v} + \frac{(1-\mu^2)}{v}\frac{\partial f}{\partial \mu}\right)$$
$$= n_e \Gamma_e \left[\frac{(1+z_i)}{v^3}\mathcal{L} f + \frac{1}{v^2}\frac{\partial}{\partial v}\left(f + \frac{1}{2}\frac{v_e^2}{v}\frac{\partial f}{\partial v}\right)\right] \tag{110}$$

where the axis of polar coordinates has been chosen to lie in the direction of the electric field. By rewriting this equation in terms of the dimensionless velocity v/v_e and the dimensionless time t/τ_c, where $\tau_c^{-1} = n_e \Gamma_e / v_e^3$, it is found that the solutions depend upon the dimensionless parameter E/E_D, where

$$E_D = m_e v_e / e\tau_c \tag{111}$$

is called the Dreicer field (Dreicer, 1960).

Note that the collision terms become relatively small at high velocity. In particular, the angle-scattering rate $D_\perp / v^2$ goes as v^{-3}, while the electric field term goes roughly as v^{-1}. The electric field term thus dominates for $v \gg v_R$, where

$$v_R \equiv (E/E_D)^{-1/2} v_e \tag{112}$$

is the runaway velocity.

The runaway of a single test electron is described by (61), augmented by the electric field acceleration term. The rate of change of the test electron's kinetic

energy is given by

$$\frac{d}{dt}\left(\frac{1}{2} m_e u^2\right) = -eE\cdot u - \nu_E\left(\frac{1}{2} m_e u^2\right). \tag{113}$$

Since the energy-loss rate (92) can be approximated by

$$\nu_E \simeq 2 n_e \Gamma_e / v^3$$

it is clear that the kinetic energy increases indefinitely for $u > v_R$, and that for $u \gg v_R$ there is essentially free acceleration of the electron.

Much analytical and computational work has been done on this problem; see the paper by Cohen (1976) and the papers cited therein. The goal of these efforts was to calculate the runaway rate γ_R (as a function of E/E_D) which is defined as follows. Integrating (110) over all velocity out to a maximum velocity v_m gives

$$\frac{d}{dt}\int_{-1}^{1} d\mu \int_0^{v_m} v^2 dv\, f(\mu, v) = \frac{e}{m_e} E v_m^2 \int_{-1}^{1} \mu\, d\mu\, f(\mu, v_m)$$
$$\equiv -\gamma_R \int_{-1}^{1} d\mu \int_0^{v_m} v^2 dv\, f(\mu, v)$$

where the velocity v_m has been taken large enough that the collisional contribution is negligible. For $v_m \gg v_R$, it is found that γ_R becomes independent of v_m. For $E/E_D \lesssim 0.2$, and for $z_i = 1$, Kulsrud et al. (1973) found

$$\gamma_R \simeq 0.35\varepsilon^{-3/8}\exp\{-[(2/\varepsilon)^{1/2} + (1/4\varepsilon)]\} \tag{114}$$

where

$$\varepsilon = E/2E_D.$$

The non-vanishing of γ_R means that some surface terms, obtained by integration by parts in velocity, are actually not zero when $E \neq 0$. Some care is therefore needed when using moments of the Fokker–Planck equation when an electric field is present, although the terms usually neglected are, in fact, exponentially small for small E/E_D.

Approximations to the Fokker–Planck collision terms

Small mass-ratio approximations. Approximate forms for the Fokker–Planck collision terms C_{ab} and C_{ba} which are simpler and more tractable in applications can be obtained when

$$m_a / m_b \ll 1.$$

In the equation (14) for C_{ab}, the term involving h_b is negligible because of the small factor m_a/m_b, so only g_b defined by (9) needs to be evaluated. By using

$$|v - v'| = |v - u_b - (v' - u_b)| \simeq |v - u_b|,$$

an approximation which is valid for velocities v (of the a species) and v' (of the b

species) such that

$$|\boldsymbol{v}-\boldsymbol{u}_b| \sim v_a \gg |\boldsymbol{v}'-\boldsymbol{u}_b| \sim v_b,$$

where $\boldsymbol{u}_b$ is the mean velocity of species b,

$$\frac{\partial^2 g_b}{\partial \boldsymbol{v}\partial \boldsymbol{v}} \simeq n_b\left(\frac{\boldsymbol{I}}{w}-\frac{\boldsymbol{w}\boldsymbol{w}}{w^3}\right) \tag{115}$$

where

$$\boldsymbol{w}=\boldsymbol{v}-\boldsymbol{u}_b.$$

Hence,

$$C_{ab}[f_a, f_b] \simeq \frac{1}{2}\Gamma_a n_b z_b^2 \frac{\partial}{\partial \boldsymbol{w}}\cdot\left[\left(\frac{\boldsymbol{I}}{w}-\frac{\boldsymbol{w}\boldsymbol{w}}{w^3}\right)\cdot\frac{\partial f_a}{\partial \boldsymbol{w}}\right], \tag{116}$$

which represents isotropic scattering of a species particles in the reference frame moving relative to the laboratory frame with the mean velocity $\boldsymbol{u}_b$ of the particles of species b. The next correction to (116) is of order m_a/m_b; its form is usually not needed explicitly, although its energy moment, calculated with f_a and f_b Maxwellians, may be obtained from (52).

Equation (116) may be linearized by assuming f_a and f_b to be close to Maxwellian, with

$$f_a = f_{a0}+f_{a1} \tag{117}$$

where

$$f_{a0}=\frac{n_a}{\pi^{3/2}v_a^3}\exp(-v^2/v_a^2) \tag{118}$$

and

$$|f_{a1}/f_{a0}| \ll 1 \tag{119}$$

(and similarly for f_b). Thus, $\boldsymbol{u}_b$ is assumed to be a first-order quantity, with

$$|u_b/v_a| \ll 1. \tag{120}$$

Neglecting terms quadratic in f_{a1}, f_{b1} and $\boldsymbol{u}_b$, C_{ab} becomes

$$C_{ab}^{\ell}(f_{a1}; f_{b1}) = \frac{n_b z_b^2 \Gamma_a}{2v^3}\left[\frac{\partial}{\partial \boldsymbol{v}}\cdot(v^2\boldsymbol{I}-\boldsymbol{v}\boldsymbol{v})\cdot\frac{\partial f_{a1}}{\partial \boldsymbol{v}}+\frac{m_a}{T_a}\boldsymbol{v}\cdot\boldsymbol{u}_b f_{a0}\right] \tag{121}$$

where

$$n_b\boldsymbol{u}_b = \int d^3v\, \boldsymbol{v} f_{b1} \tag{122}$$

which is a linear operator acting on f_{a1} and f_{b1}. The momentum transfer rate obtained from this linearized collision operator is

$$\boldsymbol{F}_{ab} = \frac{m_a n_a \boldsymbol{u}_b}{\tau_{ab}} - m_a n_b z_b^2 \Gamma_a \int d^3v \frac{\boldsymbol{v}}{v^3} f_{a1}(\boldsymbol{v}) \tag{123}$$

where τ_{ab} is given by (47), with terms of order m_a/m_b neglected.

The small m_a/m_b limit of C_{ba} is obtained in a similar way. Starting with

$$C_{ba}[f_b, f_a] = -z_a^2\Gamma_b\frac{\partial}{\partial \boldsymbol{v}}\cdot\left(\frac{m_b}{m_a}\frac{\partial h_a}{\partial \boldsymbol{v}}f_b - \frac{1}{2}\frac{\partial^2 g_a}{\partial \boldsymbol{v}\partial \boldsymbol{v}}\cdot\frac{\partial f_b}{\partial \boldsymbol{v}}\right) \tag{124}$$

approximate expressions are needed for $\partial h_a/\partial \boldsymbol{v}$ and $\partial^2 g_a/\partial \boldsymbol{v}\partial \boldsymbol{v}$. The former is obtained from the definition of h_a, (10), by using $\boldsymbol{u}=\boldsymbol{v}-\boldsymbol{v}'$ as the integration variable and expanding f_a in a Taylor series, assuming

$$v \sim v_b \ll u \simeq v' \sim v_a.$$

Thus,

$$\frac{\partial h_a}{\partial \boldsymbol{v}} \simeq \int \frac{d^3u}{u}\frac{\partial f_a}{\partial \boldsymbol{u}} + \boldsymbol{v}\cdot\int\frac{d^3u}{u}\frac{\partial^2 f_a}{\partial \boldsymbol{u}\partial \boldsymbol{u}}. \tag{125}$$

Similarly,

$$\frac{\partial^2 g_a}{\partial \boldsymbol{v}\partial \boldsymbol{v}} \simeq \int d^3v' f_a(\boldsymbol{v}')\left(\frac{\boldsymbol{I}}{v'}-\frac{\boldsymbol{v}'\boldsymbol{v}'}{(v')^3}\right). \tag{126}$$

The linearized form of (124) is obtained by using (117)–(119) and neglecting quadratic terms, as before:

$$C_{ba} \simeq C_{ba}[f_{b0}, f_{a0}] + C_{ba}^{\ell} \tag{127}$$

where

$$C_{ba}[f_{b0}, f_{a0}] = \frac{m_a n_a}{m_b n_b \tau_{ab}}\left(1-\frac{T_a}{T_b}\right)\frac{\partial}{\partial \boldsymbol{v}}\cdot(\boldsymbol{v} f_{b0}) \tag{128}$$

with τ_{ab} defined by (47) and

$$C_{ba}^{\ell} = \frac{1}{m_b n_b}\left(\boldsymbol{F}_{ab}-\frac{m_a n_a \boldsymbol{u}_b}{\tau_{ab}}\right)\cdot\frac{\partial f_{b0}}{\partial \boldsymbol{v}} + \frac{m_a n_a}{m_b n_b \tau_{ab}}\frac{\partial}{\partial \boldsymbol{v}}\cdot\left[\boldsymbol{v} f_{b1}+\frac{T_a}{m_b}\frac{\partial f_{b1}}{\partial \boldsymbol{v}}\right] \tag{129}$$

where $\boldsymbol{F}_{ab}$ is given by (123). In the terms depending on f_{a1}, the second term in (125) and the expression in (126) have been neglected, since they are generally smaller by $(m_a/m_b)^{1/2}$.

The linearized Fokker–Planck collision operators. In transport theory applications, the Fokker–Planck collision terms (21) are usually linearized by assuming the distribution functions to be close to Maxwellians, as in (117)–(119). Substituting (117) and a similar expression for f_b into (21), neglecting terms quadratic in f_{a1} and f_{b1} gives

$$C_{ab}[f_{a0}+f_{a1}, f_{b0}+f_{b1}] \simeq C_{ab}[f_{a0}, f_{b0}] + C_{ab}^{\ell} \tag{130}$$

where the first term on the right-hand side is zero if both Maxwellians have the same temperature.

The second term on the right-hand side of (130) is the linearized collision operator

$$C_{ab}^{\ell} \equiv C_{ab}[f_{a1}, f_{b0}] + C_{ab}[f_{a0}, f_{b1}] \tag{131}$$

$$= \frac{c_{ab}}{m_a}\frac{\partial}{\partial v}\cdot\int d^3v' f_{a0}(v) f_{b0}(v')\left(\frac{I}{u}-\frac{uu}{u^3}\right) \cdot\left[\frac{1}{m_a}\frac{\partial \hat{f}_{a1}}{\partial v} - \frac{1}{m_b}\frac{\partial \hat{f}_{b1}}{\partial v'} + \frac{(v+v')}{2}\left(\hat{f}_{a1}(v)+\hat{f}_{b1}(v')\right)\left(\frac{1}{T_b}-\frac{1}{T_a}\right)\right] \tag{132}$$

where $\hat{f}_{a1} \equiv f_{a1}/f_{a0}$ and $\hat{f}_{b1} \equiv f_{b1}/f_{b0}$. The first term on the right-hand side of (131) is a differential operator acting on f_{a1}, while the second term is an integral operator acting on f_{b1}. The differential operator represents the effect of collisions with a Maxwellian background of particles of species b and is the same as the collision operator in the test-particle form of the Fokker–Planck equation, (94). The integral operator in (131) represents the effect of the perturbation in the distribution function for the "field particles" of species b. In transport theory applications (132) is used only when $T_b = T_a$. Only if $m_a/m_b \ll 1$ is it permissible to take $T_b \neq T_a$, and then the small mass-ratio approximations given in the previous subsection are more convenient to use.

The linearized collision operator has most of the same properties as the nonlinear collision term, given in the section on the Fokker–Planck equation. These properties are used quite often in transport theory.

Since (132) still has the form (18), we still have particle conservation:

$$\int d^3v\, C_{ab}^{\ell} = 0. \tag{133}$$

Note that the integrand in (132) simply changes sign upon making the replacements $a \leftrightarrow b$, $v \leftrightarrow v'$. The momentum and energy conservation properties are therefore obtained in the same way as before:

$$\int d^3v\, m_a v C_{ab}^{\ell} + \int d^3v\, m_b v C_{ba}^{\ell} = 0 \tag{134}$$

$$\int d^3v \frac{m_a v^2}{2} C_{ab}^{\ell} + \int d^3v \frac{m_b v^2}{2} C_{ba}^{\ell} = 0 \tag{135}$$

If $T_a = T_b$, the H-theorem proof can be used to show the following: the linearized collision operator is zero,

$$\sum_b C_{ab}^{\ell} = 0 \tag{136}$$

if and only if f_{a1} has the form

$$f_{a1} = m_a\left(\alpha_a + \beta\cdot v + v^2/2\right) f_{a0} \tag{137}$$

(and similarly for f_{b1}). This is the perturbation of a Maxwellian distribution function due to perturbations in density, mean velocity, and temperature, Taylor expanded to first order in these perturbations.

One further property of the linearized Fokker–Planck collision operators is of interest: self-adjointness. For the like-species operator, we have

$$\int d^3v\, \hat{g}_{a1} C_{aa}^{\ell} \hat{f}_{a1} = \int d^3v\, \hat{f}_{a1} C_{aa}^{\ell} \hat{g}_{a1} \tag{138}$$

while for the unlike-species operators, this property takes the form

$$\sum_{a,b} \int d^3v\, \hat{g}_{a1} C_{ab}^{\ell}(\hat{f}_{a1}; \hat{f}_{b1}) = \sum_{a,b} \int d^3v\, \hat{f}_{a1} C_{ab}^{\ell}(\hat{g}_{a1}; \hat{g}_{b1}). \tag{139}$$

These properties are easily demonstrated, starting from (132) with $T_a = T_b$, integrating by parts, and symmetrizing with respect to the integration variables and the species subscripts, as in the section on the H-theorem.

Similarly, it can be shown that the differential and integral operators in (131) are separately self-adjoint in the following sense:

$$\int d^3v\, \hat{g}_{a1} C_{ab}[f_{a1}, f_{b0}] = \int d^3v\, \hat{f}_{a1} C_{ab}[g_{a1}, f_{b0}] \tag{140}$$

$$\int d^3v\, \hat{g}_{a1} C_{ab}[f_{a0}, f_{b1}] = \int d^3v\, \hat{f}_{b1} C_{ba}[f_{b0}, g_{a1}]. \tag{141}$$

These relations are used in the transport theory applications described in the section on ion parallel transport.

1.5.3. Classical transport processes

Spatial transport due to velocity-space diffusion in the presence of spatial gradients in the plasma, as well as electric and magnetic fields, is now considered. The Fokker–Planck equation, including these terms, is

$$\frac{\partial f_a}{\partial t} + v\cdot\nabla f_a + \frac{e_a}{m_a}\left(E + \frac{v}{c}\times B\right)\cdot\frac{\partial f_a}{\partial v} = \sum_b C_{ab} \tag{142}$$

where E and B are the electric and magnetic fields, and C_{ab} is the Fokker–Planck collision term. The collision term is assumed to be still given by (14), even when these gradients and fields are present.

Among the phenomena described by (142) are classical transport processes, which are spatially local, i.e. the fluxes of particles and energy are due to forces at approximately the same spatial location. In order for this situation to occur, the plasma must be collision dominated, with mean free paths much shorter than the gradient length in the direction of the magnetic field. In addition, either the mean free path or the mean gyroradius must be much shorter than the gradient lengths in the directions perpendicular to the magnetic field. The particles near a given point can then be affected only by the forces within a mean free path or a gyroradius. It is assumed here that the mean gyroradius is shorter that the mean free path, and is the particle localization distance perpendicular to the magnetic field.

When the transport processes are local, the plasma may be considered to be made up of many approximately closed subsystems, with slightly different densities, mean velocities and temperatures. Charged-particle collisions tend to force each subsystem to local thermodynamic equilibrium, with the subsystem entropies being maximized, subject to the constraints imposed by particle, momentum and energy conservation. Because of the small differences between subsystems, the velocity distributions for these subsystems depart slightly from Maxwellians. For example, the distribution of the velocity component in the direction of the temperature gradient is skewed somewhat in the direction of motion of those particles coming from the hotter region. As a result, there are small fluxes of particles, momentum and energy between subsystems, which are approximately linear in the thermodynamic forces (e.g. the density and temperature gradients). The resulting entropy fluxes between subsystems then make the plasma as a whole tend towards a state of global thermal equilibrium. Because of the boundary conditions and other external constraints, such as applied electromotive forces, the plasma generally is not able to reach this equilibrium state but remains in a nonequilibrium steady state. The charged particles and energy are lost from the plasma at the same rate that they are produced in the plasma in this steady state. It is the goal of transport theory to calculate these loss rates, assuming they are due to Coulomb collisions.

Basic expansion procedure

The assumed smallness of the mean free path and the mean gyroradius is exploited mathematically by using a procedure very similar to the Chapman–Enskog method in the kinetic theory of gases (Chapman and Cowling, 1952). The distribution functions are expanded in powers of the small parameter ε, where

$$\varepsilon = \lambda / l_{\|} \sim \rho / l_{\perp} \sim E/E_{\mathrm{D}}.$$

Here λ is the mean free path, ρ is the mean gyroradius, $l_{\|}$ and $l_{\perp}$ are the parallel and perpendicular gradient lengths, E is the electric field, and E_{D} is the Dreicer field, defined by (111). Thus,

$$f_a = f_{a0} + f_{a1} + f_{a2} + \cdots \tag{143}$$

where the numerical subscript denotes the power of ε. In the Fokker–Planck equation, (142), the gradient and electric field terms are considered to be first order in ε, while the time derivative is second order in ε, compared with the magnetic field and collision terms, which are considered to be comparable. The zeroth-order terms are

$$\frac{e_a}{m_a}\frac{\boldsymbol{v}}{c} \times \boldsymbol{B} \cdot \frac{\partial f_{a0}}{\partial \boldsymbol{v}} = \sum_b C_{ab}(f_{a0}, f_{b0}). \tag{144}$$

That the only solutions of this equation are Maxwellians is easily demonstrated, as follows. Multiplying by $\ln f_{a0}$, integrating over all velocity, and summing over a yields:

$$\sum_{a,b} \int \mathrm{d}^3 v \ln f_{a0} C_{ab} = 0 \tag{145}$$

which, according to (38), implies that f_{a0} has the form

$$f_{a0} = \left(n_a / \pi^{3/2} v_a^3\right) \exp\left(-v^2/v_a^2\right) \tag{146}$$

where

$$v_a^2 = 2T_a / m_a \tag{147}$$

and $T_a = T$ for all a. The common mean velocity of all particle species has been chosen to be zero. Although all of the temperatures are required to be equal at this point, they will continue to be distinguished by species subscripts for later convenience.

The first-order terms from (142) give

$$\sum_b \left[C_{ab}(f_{a1}, f_{b0}) + C_{ab}(f_{a0}, f_{b1})\right] - \Omega_a \boldsymbol{v} \times \hat{\boldsymbol{n}} \cdot \frac{\partial f_{a1}}{\partial \boldsymbol{v}} = \left[\boldsymbol{v} \cdot \left(\frac{\nabla p_a}{p_a} - \frac{e_a}{T_a}\boldsymbol{E}\right) + \left(\frac{v^2}{v_a^2} - \frac{5}{2}\right)\frac{\boldsymbol{v} \cdot \nabla T_a}{T_a}\right] f_{a0} \tag{148}$$

where $\Omega_a = e_a B / m_a c$ is the gyrofrequency, $\hat{\boldsymbol{n}} = \boldsymbol{B}/B$ is a unit vector tangent to the magnetic field, and $p_a = n_a T_a$ is the partial pressure of species a. Equation (148) is a system of linear integrodifferential equations for the functions f_{a1}, and the primary mathematical problem in classical transport is to solve these equations.

Once the solution of (148) is obtained, the first-order particle flux, defined by

$$n_a \boldsymbol{u}_{a1} = \int \mathrm{d}^3 v\, \boldsymbol{v} f_{a1}, \tag{149}$$

may be calculated. The heat flux is defined as the flux of energy in excess of that transported by mass motion:

$$\boldsymbol{q}_a \equiv \int \mathrm{d}^3 v \frac{1}{2} m (\boldsymbol{v} - \boldsymbol{u}_a)^2 (\boldsymbol{v} - \boldsymbol{u}_a) f_a. \tag{150}$$

By using (143) and retaining the lowest-order nonzero terms, which are first order in ε, the first-order heat flux is obtained:

$$\boldsymbol{q}_{a1} = \int \mathrm{d}^3 v \left(\frac{1}{2} m_a v^2 - \frac{5}{2} T_a\right) \boldsymbol{v} f_{a1}. \tag{151}$$

Expressions may be obtained for the component of the particle flux perpendicular to the magnetic field, by multiplying (148) by $m_a \boldsymbol{v}$ and integrating over all velocities:

$$\sum_b \int \mathrm{d}^3 v\, m_a \boldsymbol{v} \left[C_{ab}(f_{a1}, f_{b0}) + C_{ab}(f_{a0}, f_{b1})\right] + \frac{n_a e_a}{c} \boldsymbol{u}_{a1} \times \boldsymbol{B} = \nabla p_a - n_a e_a \boldsymbol{E}. \tag{152}$$

By using the definition of the first-order friction force,

$$F_{ab} = \int d^3v\, m_a v[C_{ab}(f_{a1}, f_{b0}) + C_{ab}(f_{a0}, f_{b1})] \tag{153}$$

and solving for the flux,

$$n_a u_{a\perp} = n_a \frac{c}{B^2} E \times B + \frac{c}{e_a B^2} B \times \left(\nabla p_a - \sum_{b(\neq a)} F_{ab} \right). \tag{154}$$

Note that the friction force is not a known quantity since it can only be calculated after (148) has been solved for f_{a1} and f_{b1}.

A similar expression for the perpendicular heat flux is obtained by multiplying (148) by $m_a v(v^2/v_a^2 - 5/2)$, integrating over all velocities, and solving for $q_{a\perp}$:

$$q_{a\perp} = \frac{5}{2} \frac{cp_a}{e_a B^2} B \times \nabla T_a + \frac{cT_a}{e_a B^2} \sum_b G_{ab} \times B \tag{155}$$

where the heat friction vector is defined by

$$G_{ab} \equiv \int d^3v \left(\frac{v^2}{v_a^2} - \frac{5}{2} \right) m_a v[C_{ab}(f_{a1}, f_{b0}) + C_{ab}(f_{a0}, f_{b1})]. \tag{156}$$

Equation (155), like (154), will be used in the next subsection to obtain expressions for the fluxes in terms of known quantities. In their present form, these equations can be used to indicate the physical origins of the fluxes.

In (154), the first term is due to the $E \times B$ drift motion of the particle guiding centers. The second term is due to the diamagnetism of the plasma, which has its microscopic origin in the gyration of the particles around their guiding centers. The third term is the collisional transport flux perpendicular to the magnetic field. The transport is in a direction perpendicular to the total friction force. Note that, since $F_{aa} = 0$, like-species collisions make no direct contribution to the transport; thus the $b = a$ term is omitted in (154). If the particle flux were calculated to third order in the expansion parameter ε, there would be a contribution from like-species collisions, in the form of viscous-stress terms (Simon, 1955; Kaufman, 1958). These are beyond the scope of the present article.

Equation (154) may be multiplied by e_a and summed over species to obtain the perpendicular current density:

$$j_\perp = \sum_a n_a e_a u_{a\perp} = \frac{c}{B^2} B \times \nabla P \tag{157}$$

where

$$P = \sum_a p_a$$

is the total pressure. The friction terms have cancelled out, because of momentum conservation, (25). Thus, the collisional transport is ambipolar and gives rise to no net current density. Also, the electric field terms have cancelled because it has been assumed that the plasma is electrically neutral:

$$\sum_a n_a e_a = 0. \tag{158}$$

Note also that the component of (152) parallel to the magnetic field is

$$\hat{n} \cdot \sum_b F_{ab} = \hat{n} \cdot \nabla p_a - n_a e_a \hat{n} \cdot E. \tag{159}$$

Summing over all species and using momentum conservation (25), and charge neutrality (158), gives

$$\hat{n} \cdot \nabla P = 0. \tag{160}$$

Equations (157) and (160) are equivalent to the MHD equilibrium force-balance equation

$$(1/c) j \times B = \nabla P. \tag{161}$$

In (155), the first term is the heat flux due to particles gyrating around their guiding centers, while the second term is the collisional heat transport. Note that like-species collisions, as well as unlike-species collisions, contribute directly to the heat flux.

Returning to the expansion of (142) in powers of ε: the second-order terms give

$$\sum_b [C_{ab}(f_{a2}, f_{b0}) + C_{ab}(f_{a1}, f_{b1}) + C_{ab}(f_{a0}, f_{b2})] - \Omega_a v \times \hat{n} \cdot \frac{\partial f_{a2}}{\partial v}$$
$$= \frac{\partial f_{a0}}{\partial t} + v \cdot \nabla f_{a1} + \frac{e_a}{m_a} E \cdot \frac{\partial f_{a1}}{\partial v}. \tag{162}$$

Integrating this equation over all velocities, and making use of the particle-conserving property of the collision terms yields:

$$\partial n_a / \partial t + \nabla \cdot (n_a u_{a1}) = 0 \tag{163}$$

where $n_a u_{a1}$ is the first-order particle flux, defined by (149). First multiplying (162) by $\frac{1}{2} m_a v^2$ and then integrating over velocity yields the energy-balance equation

$$\frac{\partial}{\partial t}\left(\frac{3}{2} n_a T_a \right) + \nabla \cdot \left(q_{a1} + \frac{5}{2} n_a T_a u_{a1} \right) = n_a e_a u_{a1} \cdot E + \sum_b (Q_{ab} + u_a \cdot F_{ab}) \tag{164}$$

where q_{a1} is the first-order heat flux defined by (151) and Q_{ab} is the energy-transfer rate defined by (28). Note that Q_{ab} is not zero, because of the contributions from f_{a2} and f_{b2}, in (162). In order to eliminate these unknown contributions, we sum (164) over species. We now also make use of the equality, for all species, of the temperatures contained in the Maxwellian zero-order distributions, (146). Thus,

$$(\partial / \partial t)\left(\frac{3}{2} nT \right) + \nabla \cdot Q = j \cdot E \tag{165}$$

where $n = \sum_a n_a$ is the total number density, T is the common temperature of all particle species, Q is the energy flux defined by

$$Q=\sum_a\left(q_{a1}+\frac{5}{2}Tn_a u_{a1}\right) \tag{166}$$

and j is the current density:

$$j=\sum_a n_a e_a u_{a1}. \tag{167}$$

The collisional conservation of energy, (33), has been used to eliminate the collisional terms.

The goal of transport theory is to obtain expressions for the particle and heat fluxes in terms of gradients of the number densities, n_a, and the temperature T, so that (163) and (165) might then be used to determine the time and space dependence of the densities and the temperature.

The strong magnetic field limit

Transport in MHD equilibrium systems has been considered. The most important examples of such systems are strongly magnetized, in the sense that

$$\Omega_a\tau_{aa}\gg 1. \tag{168}$$

This is the strong magnetic field limit, in which the particle gyration time $2\pi/\Omega_a$ is much shorter than the mean collision time τ_{aa}, and consequently the mean gyroradius $\rho_a=v_a/\Omega_a$ is much smaller than the mean free path, $\lambda_a=v_a\tau_{aa}$.

In order to investigate this limit, use will be made of the cylindrical velocity coordinates $v_\parallel, v_\perp, \zeta$ defined by

$$v=v_\parallel\hat{n}+v_\perp(\hat{e}_1\cos\zeta+\hat{e}_2\sin\zeta)\equiv v_\parallel\hat{n}+v_\perp \tag{169}$$

where $\hat{n}=B/B$ is a unit vector tangent to the magnetic field, and $\hat{e}_1, \hat{e}_2$ are two other mutually perpendicular unit vectors. The first-order distribution function is written as the sum of two terms

$$f_{a1}=\bar{f}_{a1}+\tilde{f}_{a1} \tag{170}$$

where $\bar{f}_{a1}$ is the average over the gyration angle

$$\bar{f}_{a1}\equiv\oint\frac{d\zeta}{2\pi}f_{a1} \tag{171}$$

The average over ζ of (148) is

$$\sum_b\left[C_{ab}(\bar{f}_{a1},f_{b0})+C_{ab}(f_{a0},\bar{f}_{b1})\right]$$
$$=v_\parallel\left[\left(\frac{\nabla_\parallel p_a}{p_a}-\frac{e_a}{T_a}E_\parallel\right)+\left(\frac{v^2}{v_a^2}-\frac{5}{2}\right)\frac{\nabla_\parallel T_a}{T_a}\right]f_{a0} \tag{172}$$

where $\nabla_\parallel$ and $E_\parallel$ are the parallel (to B) components of the gradient and electric field, respectively. This equation determines the transport parallel to the magnetic field, and will be considered in the section on parallel transport. Subtracting (172) from (148) gives

$$\sum_b\left[C_{ab}(\tilde{f}_{a1},f_{b0})+C_{ab}(f_{a0},\tilde{f}_{b1})\right]+\Omega_a\frac{\partial\tilde{f}_{a1}}{\partial\zeta}$$
$$=v_\perp\cdot\left[\left(\frac{\nabla p_a}{p_a}-\frac{e_a}{T_a}E\right)+\left(\frac{v^2}{v_a^2}-\frac{5}{2}\right)\frac{\nabla T_a}{T_a}\right]f_{a0} \tag{173}$$

where $v_\perp$ is defined by (169).

The assumption expressed by (168) is now used and $\tilde{f}_{a1}$ is expanded in powers of $(\Omega_a\tau_{aa})^{-1}$:

$$\tilde{f}_{a1}=\tilde{f}_{a1}^{(0)}+\tilde{f}_{a1}^{(1)}+\cdots \tag{174}$$

In the zeroth-order version of (173), the collision terms are absent, so it may be integrated immediately, giving

$$\tilde{f}_{a1}^{(0)}=\frac{v\times\hat{n}}{\Omega_a}\cdot\left[\left(\frac{\nabla p_a}{p_a}-\frac{e_a}{T_a}E\right)+\left(\frac{v^2}{v_a^2}-\frac{5}{2}\right)\frac{\nabla T_a}{T_a}\right]f_{a0}. \tag{175}$$

Substituting (175) into (149) gives the first two terms in (154). Substituting into (151) gives the first term of (155). In order to obtain the collisional terms in (154) and (155), the friction force F_{ab} and the heat friction vector G_{ab} must be evaluated. This can be done by approximating the first-order distribution function, using (175). Note that this expression, which is zeroth order in the collision frequency, is sufficient to obtain the collisional fluxes to first order, by using (154) and (155) in which the collision terms appear explicitly in F_{ab} and G_{ab}. Thus, it is not necessary to calculate $\tilde{f}_{a1}^{(1)}$.

By substituting (175) (and the similar expression for f_{b1}) into (153), the integrals can be carried out. Using (132) for C_{ab}^l with $T_a=T_b=T$ gives

$$F_{ab}=\frac{m_a n_a}{\tau_{ab}}\frac{c}{eB}\hat{n}\times\left(\frac{\nabla p_b}{z_b n_b}-\frac{\nabla p_a}{z_a n_a}+\frac{3}{2z_a}\frac{(1-z_a m_a/z_b m_b)}{(1+m_a/m_b)}\nabla T\right). \tag{176}$$

Hence, by substituting into (154),

$$n_a u_{a_\perp}^{(1)}\equiv\frac{-c}{e_a B}\hat{n}\times\sum_{b(\neq a)}F_{ab}$$
$$=\frac{n_a m_a c^2}{e_a^2 B^2}\sum_{b(\neq a)}\tau_{ab}^{-1}\left(\frac{z_a}{z_b}\frac{\nabla_\perp p_b}{n_b}-\frac{\nabla_\perp p_a}{n_a}+\frac{3}{2}\frac{(1-z_a m_a/z_b m_b)}{(1+m_a/m_b)}\nabla_\perp T\right). \tag{177}$$

Similar calculations enable G_{ab}, defined by (156), to be evaluated and then (155) gives

$$q_{a\perp}^{(1)} \equiv \frac{-cT}{e_a B}\hat{n} \times \sum_b G_{ab}$$

$$= \frac{n_a m_a c^2 T}{e_a^2 B^2} \sum_b \frac{1}{\tau_{ab}(1+m_a/m_b)} \left\{ \frac{3}{2}\left(\frac{\nabla_\perp p_a}{n_a} - \frac{z_a}{z_b}\frac{\nabla_\perp p_b}{n_b} \right) - \left[\frac{15}{2}\left(\frac{m_a}{m_b}\right)^2 + 4\frac{m_a}{m_b} + \frac{13}{4} - \frac{27}{4}\frac{m_a z_a}{m_b z_b} \right] \frac{\nabla_\perp T}{(1+m_a/m_b)} \right\}. \quad (178)$$

The individual terms in the summations in (177) and (178) may be interpreted in terms of random walks with step size $\rho_a = v_a/\Omega_a$, the mean gyroradius, taken at intervals of τ_{ab}, the momentum transfer collision time. This is the time for an average particle to diffuse in velocity through an angle of roughly 90° if there were no magnetic field. In a strong magnetic field, the velocity diffusion manifests itself through diffusion of the position of the guiding center, $r_g = r + (v \times \hat{n})/\Omega_a$, where r and v are the position and velocity of the particle itself. Thus, the diffusion coefficient, the factor multiplying $\nabla n_a - (z_a n_a / z_b n_b)\nabla n_b$ in (177), is

$$D = \tfrac{1}{2}\rho_a^2/\tau_{ab}.$$

The pressure gradients appear in (177) in those particular combinations because the collisional flux is proportional to the magnitude of the friction force, which in turn depends on the difference in the zeroth-order mean velocities of the two species involved:

$$u_{b\perp}^{(0)} - u_{a\perp}^{(0)} = \frac{c}{B^2} B \times \left(\frac{\nabla p_b}{e_b n_b} - \frac{\nabla p_a}{e_a n_a} \right). \quad (179)$$

In the absence of a temperature gradient, the diffusion due to collisions between two species a and b would stop if their densities were related according to

$$n_b^{z_a}/n_a^{z_b} = \text{constant},$$

since the zeroth-order mean velocities would be equal in that case. For a simple plasma, with $z_a = -1$ for electrons, this cannot occur unless the electron density is constant, assuming the plasma to be neutral. If a and b are two ionic species, this can occur if the more highly-charged species (the "impurity") is more highly concentrated in the center of the plasma than the other ionic species. If this relation between the density gradients is not satisfied, then diffusion proceeds in such a direction as to satisfy it. This generally means, for the hypothetical case of only one impurity species, that impurities diffuse into the plasma.

The appearance of the temperature gradient in (177), called the Nernst effect, is due to the fact that temperature gradients give rise to friction forces, as a result of the dependence of the collision frequency on particle energy. The contributions to the momentum transfer from neighboring regions of different temperature are therefore different, leaving a net momentum transfer, and hence a net particle flux. For example, if the density gradients were zero then the lighter of the two species would diffuse towards the higher-temperature region as a result of collisions between the species, assuming

$$m_a/m_b < z_a/z_b \leqslant 1.$$

The direction of this thermal diffusion may be reversed, however, if these inequalities are not satisfied.

A similar effect, the Ettingshausen effect, is the appearance of the pressure gradient terms in (178) for the heat flux. The faster electrons, for example, diffuse more slowly in the presence of pressure gradients than the slower ones. This appears as a flux of heat in the direction opposite to the particle flux when the temperature gradient is zero.

The results obtained so far for the particle and heat fluxes may now be summarized as follows. The particle flux is

$$n_a u_a = n_a u_{a\perp}^{(0)} + n_a u_{a\perp}^{(1)} + n_a \hat{n} u_{a\parallel} \quad (180)$$

where

$$n_a u_{a\perp}^{(0)} = n_a \frac{c}{B^2} E \times B + \frac{c}{e_a B^2} B \times \nabla p_a. \quad (181)$$

$n_a u_{a\perp}^{(1)}$ is given by (177), and $u_{a\parallel}$ is still to be determined. The first two terms in (180) are both first order in the small gyroradius parameter $\varepsilon = \rho/l_\perp$; the first is zeroth order in $(\Omega_a \tau_{aa})^{-1}$, the ratio of collision frequency to gyrofrequency, while the second term is first order in $(\Omega_a \tau_{aa})^{-1}$. The heat flux is

$$q_a = q_{a\perp}^{(0)} + q_{a\perp}^{(1)} + \hat{n} q_{a\parallel} \quad (182)$$

where

$$q_{a\perp}^{(0)} = \frac{5}{2}\frac{c p_a}{e_a B^2} B \times \nabla T_a. \quad (183)$$

$q_{a\perp}^{(1)}$ is given by (178), and $q_{a\parallel}$ is still to be determined. The parallel components of the particle and heat fluxes,

$$n_a u_{a\parallel} \equiv \hat{n}\cdot(n_a u_{a1}) \quad (184)$$

$$q_{a\parallel} \equiv \hat{n}\cdot q_{a1} \quad (185)$$

may be calculated, once the solution of (172) has been obtained for $\bar{f}_{a1}$:

$$n_a u_{a\parallel} = \int d^3v\, v_\parallel \bar{f}_{a1} \quad (186)$$

$$q_{a\parallel} = \int d^3v \left(\frac{1}{2} m_a v^2 - \frac{5}{2} T_a \right) v_\parallel \bar{f}_{a1}. \quad (187)$$

Rather than continue in complete generality regarding numbers and masses of species, the special case of a plasma with only one ionic species will be considered next, and a plasma with multiple ionic species will be considered later.

Transport in a simple plasma

The small mass-ratio limit. A simple plasma in which there are only two particle species, the electrons and one kind of ion, is now considered. This reduces the number of parameters such as charge ratios. It also makes possible the simplification of the unlike-species collision operators, using the smallness of the mass ratio m_e/m_i, as described in the section on small mass-ratio approximations. Since the basic expansion procedure described in the section on basic expansion procedure is somewhat different in this special case, it will now be reconsidered.

The electron version of (144) contains, on the right-hand side, the electron–ion collision term C_{ei}. This is to be approximated by using (116), neglecting corrections of order m_e/m_i. In the ion version of (149), the ion–electron collision term, C_{ie}, is to be neglected entirely. The conclusion reached above, that the zeroth-order distributions are Maxwellians with the same mean velocity, still holds, but now unequal temperatures are allowed: $T_e \neq T_i$. The analysis of the first-order equations (148) is carried out just as in the sections on basic expansion procedure and the strong magnetic field limit, except that C_{ei}^{ℓ} is approximated by (121) while C_{ie}^{ℓ} is neglected, except in (172) governing parallel transport, to be discussed in the next subsection. In the second-order equations (162), the $O(m_e/m_i)$ correction to C_{ei} is retained in the electron equation, and C_{ie} is retained in the ion equation. In both of these terms, only the zeroth-order Maxwellians are used, and only the energy moments are needed; these are given by (52), which becomes

$$Q_{ie} = \frac{3m_e}{m_i}\frac{n_e}{\tau_{ei}}(T_e - T_i) \tag{188}$$

for ions, and, from (31),

$$Q_{ei} = -Q_{ie} + (\boldsymbol{u}_i - \boldsymbol{u}_e)\cdot \boldsymbol{F}_e \tag{189}$$

for electrons. Both electron and ion versions of (164) are needed, since there are now two temperatures to be determined.

The strong magnetic field limit of perpendicular transport is obtained just as in the section the strong magnetic field limit; the perpendicular electron flux is

$$n_e\boldsymbol{u}_{e\perp} = \frac{-c}{eB}\hat{\boldsymbol{n}}\times\nabla p_e + n_e\frac{c}{B}\boldsymbol{E}\times\hat{\boldsymbol{n}} - \frac{(\nabla_\perp P - \frac{3}{2}n_e\nabla_\perp T_e)}{m_e\Omega_e^2\tau_{ei}} \tag{190}$$

where $P = p_e + p_i$ is the total pressure, while the perpendicular ion flux is

$$n_i\boldsymbol{u}_{i\perp} = \frac{c}{z_i eB}\hat{\boldsymbol{n}}\times\nabla p_i + n_i\frac{c}{B}\boldsymbol{E}\times\hat{\boldsymbol{n}} - \frac{(\nabla_\perp P - \frac{3}{2}n_e\nabla_\perp T_e)}{m_e\Omega_e^2\tau_{ei}z_i}. \tag{191}$$

The condition of charge neutrality, $n_e = z_i n_i$, has been used in both of these expressions. The perpendicular electron heat flux is

$$q_{e\perp} = -\frac{5}{2}n_e\frac{cT_e}{eB}\hat{\boldsymbol{n}}\times\nabla T_e + \frac{T_e}{m_e\Omega_e^2\tau_{ei}}\left[\frac{3}{2}\nabla_\perp P - \left(\frac{13}{4}+\frac{\sqrt{2}}{z_i}\right)n_e\nabla_\perp T_e\right] \tag{192}$$

while the perpendicular ion heat flux is

$$q_{i\perp} = \tfrac{5}{2}n_i\left(\frac{cT_i}{z_i eB}\right)\hat{\boldsymbol{n}}\times\nabla T_i - \frac{2n_iT_i\nabla_\perp T_i}{m_i\Omega_i^2\tau_{ii}}. \tag{193}$$

The electron–ion and ion–ion collision times, τ_{ei} and τ_{ii}, are defined by (56) and (54). The ion collisional heat flux exceeds the corresponding electron heat flux by roughly a factor $(m_i/m_e)^{1/2}$, although the diamagnetic terms, proportional to $\hat{\boldsymbol{n}}\times\nabla T_e$ and $\hat{\boldsymbol{n}}\times\nabla T_i$, are comparable (assuming T_e/T_i and Z_i to be comparable to unity).

Note that it is the electron temperature T_e which appears in (190) and (192); the approximation for the electron–ion collision term given by (121) does not explicitly contain the ion temperature. In deriving (193), only the ion–ion collision term was retained, so it is the ion temperature which appears. In (191), the collisional diffusion term was evaluated by using momentum conservation, in the form $\boldsymbol{F}_{ie} = -\boldsymbol{F}_{ei}$, and the result for the electron diffusion, (190).

Parallel transport. We first consider the electron version of (172),

$$C_{ee}^{\ell}f_{e1} + \nu_{ei}\left(\mathcal{L}f_{e1} + \frac{2v_\parallel u_{i\parallel}}{v_e^2}f_{e0}\right) = v_\parallel\left[\left(\frac{\nabla_\parallel p_e}{p_e} + \frac{e}{T_e}E_\parallel\right) + \left(\frac{v^2}{v_e^2} - \frac{5}{2}\right)\frac{\nabla_\parallel T_e}{T_e}\right]f_{e0}. \tag{194}$$

In this section, as well as the following sections, the overbar, which denotes the average over particle gyration angle, will be omitted for notational simplicity. Since the ion distribution function f_{i1} enters the equation only through the parallel mean velocity $u_{i\parallel}$, it can be eliminated by transforming to a reference frame in which $u_{i\parallel} = 0$. This is equivalent to making the transformation

$$f_{e1} = \frac{2v_\parallel u_{i\parallel}}{v_e^2}f_{e0} + g_{e1}. \tag{195}$$

After defining the velocity-independent forces

$$A_1 = \frac{\nabla_\parallel p_e}{p_e} + \frac{e}{T_e}E_\parallel \tag{196}$$

$$A_2 = \frac{\nabla_\parallel T_e}{T_e} \tag{197}$$

the equation for g_{e1} can be written as

$$C_{ee}^{\ell}g_{e1} + \nu_{ei}\mathcal{L}g_{e1} = v_\parallel\left[A_1 + \left(v^2/v_e^2 - \frac{5}{2}\right)A_2\right]f_{e0} \tag{198}$$

where

$$\nu_{ei}(v) = n_e z_i \Gamma_e/v^3 \tag{199}$$

and where

$$\mathfrak{L} = \frac{1}{2}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial}{\partial \mu}, \tag{200}$$

since the distribution functions which determine parallel transport are symmetric about the magnetic field direction. We note that the solution for g_{e1} must express this function as a linear combination of A_1 and A_2, since that is the form of the right-hand side of (198). The parallel current density,

$$j_{\|e} \equiv -n_e e(u_{e\|} - u_{i\|}) = -e\int d^3v\, v_{\|} g_{e1} \tag{201}$$

and the parallel electron heat flux,

$$q_{e\|} = T_e \int d^3v \left(\frac{v^2}{v_e^2} - \frac{5}{2}\right) v_{\|} g_{e1} \tag{202}$$

can be calculated from g_{e1}, and are also linear combinations of A_1 and A_2:

$$j_{\|e}/e = (n_e T_e \tau_{ei}/m_e)(\lambda_{11}A_1 + \lambda_{12}A_2) \tag{203}$$

$$-q_{e\|}/T_e = (n_e T_e \tau_{ei}/m_e)(\lambda_{21}A_1 + \lambda_{22}A_2). \tag{204}$$

Another way of expressing these results can be found as follows. By taking the appropriate moments of (194), the parallel components of the friction force and heat friction vector are given by

$$F_{e\|} = \nabla_{\|} p_e + n_e e E_{\|} = p_e A_1 \tag{205}$$

$$G_{e\|} = \tfrac{5}{2} n_e \nabla_{\|} T_e = \tfrac{5}{2} p_e A_2. \tag{206}$$

By solving the transport relations (203) and (204) for A_1 and A_2, the following inverse transport relations are found:

$$F_{e\|} = (m_e/\tau_{ei})(\mu_{11} j_{\|e}/e + \mu_{12} q_{e\|}/T_e) \tag{207}$$

$$G_{e\|} = -\left(\frac{5}{2}\frac{m_e}{\tau_{ei}}\right)(\mu_{21} j_{\|e}/e + \mu_{22} q_{e\|}/T_e) \tag{208}$$

where

$$\mu_{11} = \lambda_{22}/\Delta, \qquad \mu_{12} = \lambda_{12}/\Delta, \tag{209}$$

$$\mu_{21} = \lambda_{21}/\Delta, \qquad \mu_{22} = \lambda_{11}/\Delta \tag{210}$$

with

$$\Delta \equiv \lambda_{11}\lambda_{22} - \lambda_{21}\lambda_{12}.$$

Equation (194) was solved numerically by Spitzer and Harm (1953). Table 1 gives their results for the λ_{ij}, as well as the μ_{ij} derived from them, for $z_i = 1$, 2, 4, and 16. These results can also be obtained by a variational calculation, as described in the section on transport in a plasma with multiple ion species. The values for a Lorentz

Table 1
Spitzer–Harm transport coefficients

z_i	λ_{11}	$\lambda_{12}(=\lambda_{21})$	λ_{22}	μ_{11}	μ_{12}	μ_{22}
1	1.975	1.389	4.174	0.661	0.220	0.313
2	2.320	2.107	6.830	0.599	0.185	0.203
4	2.665	2.910	10.15	0.546	0.157	0.143
16	3.132	4.216	16.31	0.490	0.127	0.094
∞	3.395	5.093	21.22	0.460	0.110	0.074

plasma, in which $z_i \to \infty$, can be obtained easily by solving (194) analytically, neglecting the term $C_{ee}^l g_{e1}$.

The equality of λ_{12} and λ_{21}, and consequently of μ_{12} and μ_{21}, is an example of a general result from irreversible thermodynamics, the Onsager symmetry relations (de Groot and Mazur, 1962). Such relations exist when the decay of small fluctuations is governed by the macroscopic equations, including the appropriate transport processes, and when the tensor correlation function for these fluctuations has a symmetry property as a result of time-reversal invariance of the microscopic equations of motion. Microscopic reversibility manifests itself here through the self-adjointness of the collision operators, which is used to prove the Onsager relations directly, in the next subsection.

The existence of nonzero cross terms λ_{12}, λ_{21} in the transport relations is called the thermoelectric effect, and is due to the velocity dependence of the collision frequency. To clarify the origin of this effect, the transport relations are rewritten in the form of a generalized Ohm's law,

$$E_{\|} + \frac{\nabla_{\|} p_e}{n_e e} = j_{\|e}/\sigma_{\|} - (\alpha/e)\nabla_{\|} T_e \tag{211}$$

and a generalized Fourier's law,

$$q_{e\|} = -\alpha T_e j_{\|e}/e - (\lambda_{22} - \lambda_{12}^2/\lambda_{11})(p_e \tau_{ei}/m_e)\nabla_{\|} T_e \tag{212}$$

where the electrical conductivity is

$$\sigma_{\|} = \lambda_{11} n_e e^2 \tau_{ei}/m_e \tag{213}$$

and the thermoelectric coefficient is

$$\alpha = \lambda_{12}/\lambda_{11}. \tag{214}$$

The last term in (211) is analogous to the Seebeck effect in metals, the origin of the thermal EMF in thermocouples. Even in the absence of any current, when there is a temperature gradient there is a net frictional force because the electrons coming from one direction have higher energies, and hence lower collision rates, than those coming from the other direction. The first term in (212) is analogous to the Peltier effect in metals. Even in the absence of a temperature gradient, when there is electrical current heat flows because of the skewed velocity distribution which results from the velocity dependence of the collision frequency.

The thermal conductivity,

$$\kappa_{\parallel e} = (\lambda_{22} - \lambda_{12}^2/\lambda_{11})(p_e\tau_{ei}/m_e) \tag{215}$$

is positive. This follows from the negative-definite property of the collision operators, and corresponds to the positive-definite nature of entropy production, as will be demonstrated in the next subsection. The parallel electron thermal conductivity $\kappa_{\parallel e}$ exceeds the corresponding perpendicular conductivity, from (178), by roughly a factor $(\Omega_e\tau_{ei})^2$.

The ion version of (172) is now considered. The linearized ion–electron collision term, given by (129), must be retained, but in a simplified form, including only the first term, proportional to F_{ei}. This is because the perturbation in the distribution functions, and the mean velocities, for parallel transport, satisfy

$$\bar{f}_{a1}/f_{a0} \sim u_{a\parallel}/v_a \sim \lambda_a/l_\parallel \tag{216}$$

where $\lambda_a = v_a\tau_{aa}$ is roughly comparable for both electrons and ions. The ion–electron term can be neglected for perpendicular transport because

$$\tilde{f}_{e1}/f_{e0} \sim \rho_e/l_\perp \ll \rho_i/l_\perp$$

where ρ_e and ρ_i are the mean gyroradii of electrons and ions. By using (152), the friction term cancels with the ion pressure gradient and electric field terms, so that the equation for f_{i1} becomes

$$C_{ii}^{\ell} f_{i1} = v_\parallel \left(\frac{v^2}{v_i^2} - \frac{5}{2} \right) \frac{\nabla_\parallel T_i}{T_i} f_{i0}. \tag{217}$$

This equation has been solved by Braginskii (1957) using the moment method. By expanding the distribution function in Sonine polynomials, substituting into the equation, and multiplying it by Sonine polynomials and integrating, enough moment equations are generated to determine the coefficients in the distribution function. Braginskii's result for the parallel ion heat flux is

$$q_{i\parallel} = -3.91(n_iT_i\tau_{ii}/m_i)\,\nabla_\parallel T_i. \tag{218}$$

The ion parallel thermal conductivity is smaller than the electron value by roughly a factor $(m_e/m_i)^{1/2}$.

Note that the solution obtained for f_{i1} is not unique, since another solution can be obtained from it by adding a term $(2v_\parallel/v_i^2)\Delta u_\parallel f_{i0}$, corresponding to a change in the mean parallel velocity by $\Delta u_\parallel$. That the ion mean velocity $u_{i\parallel}$ is not determined is not an artifact of the small mass-ratio approximations used, but is consistent with a more general result. The center-of-mass mean velocity, in general, is not determined by the physics which has been included up to this point, which gives no net parallel force on the (neutral) plasma. By carrying out the expansion in powers of ϵ to higher order, an equation determining the time rate of change of the center-of-mass velocity would be obtained, including the effects of viscosity. These are small effects, because of the assumption which has been made in this article that the velocities themselves are small, and due to transport in MHD equilibrium systems. Since these small effects are complicated by higher-order heat conduction effects, they will not be considered in this article. Indeed, they do not seem to have been considered adequately in the literature.

The Onsager relations and the variational principle. The Onsager symmetry relations, mentioned in the previous subsection, follow from the self-adjointness of the linearized collision operators stated in the section on the linearized Fokker–Planck collision operators. For a simple plasma, the λ_{ij} which appear in (203) and (204) satisfy the symmetry relation $\lambda_{12} = \lambda_{21}$. This relation follows from the variational expressions for the λ_{ij} which will now be derived.

The solution of (198) can be expressed as a linear combination of the driving terms, A_1 and A_2:

$$g_{e1} = h_1A_1 + h_2A_2 \tag{219}$$

where A_1 and A_2 are given by (196) and (197). The functions h_1 and h_2 must be solutions of the equations

$$C_{ee}^{l} h_1 + \nu_{ei}\mathfrak{L} h_1 = v_\parallel f_{e0} \tag{220}$$

$$C_{ee}^{l} h_2 + \nu_{ei}\mathfrak{L} h_2 = v_\parallel (v^2/v_e^2 - \tfrac{5}{2}) f_{e0}. \tag{221}$$

These equations are the Euler equations for the variational principles

$$\delta(S_{jj} - 2P_{jj}) = 0, \qquad j = 1,2 \tag{222}$$

where the functionals S_{ij} and P_{ij} are defined by

$$S_{ij} = \int d^3v\, \hat{h}_i \left(C_{ee}^{\ell} h_j + \nu_{ei}\mathfrak{L} h_j \right) \tag{223}$$

and

$$P_{ij} = \int d^3v\, \hat{h}_i d_j \tag{224}$$

where

$$d_1 = v_\parallel f_{e0} \tag{225}$$

$$d_2 = v_\parallel (v^2/v_e^2 - \tfrac{5}{2}) f_{e0}. \tag{226}$$

The δ symbol here means the first variation, i.e. the difference between the functional evaluated at $h + \delta h$ and at h, neglecting terms quadratic in δh. The extremal values (in this case the maximal values) of the variational quantities are

$$S_{jj} - 2P_{jj} = -P_{jj} \tag{227}$$

which follows from (220) and (221).

These variational principles can be used to obtain approximate solutions to (220) and (221), using trial functions which depend linearly on parameters. These parameters are then determined by the solution of linear algebraic equations (Robinson and

Bernstein, 1962). For example, the simple trial function

$$h_1 = v_\|(av + bv^2)f_{e0}$$

containing only two parameters, can be used to calculate a value for λ_{11} which differs by less than one per cent from the Spitzer–Harm value, for $z_i = 1$. The good accuracy thus obtained is due to the fact that the error in a variational quantity is quadratic in the error in the trial function. Note that a direct calculation of the transport coefficient, from the definition of the flux (201), does not give good accuracy, even when a variationally-determined trial function is used.

Having obtained solutions to (220) and (221), note that the following is also a variational principle:

$$\delta(S_{12} - P_{12} - P_{21}) = 0 \tag{228}$$

where S_{12}, P_{12} and P_{21} are again defined by (223) and (224). When h_1 is varied with h_2 held fixed, this gives (221) as the Euler equation, while varying h_2 with h_1 held fixed yields (220). By using the equation for h_2, the extremal value of the variational quantity is found to be

$$S_{12} - P_{12} - P_{21} = -P_{21}. \tag{229}$$

Note, however, that $S_{12} = S_{21}$, from the self-adjointness of C^ℓ_{ee}, stated in the section on the linearized Fokker–Planck collision operators-and of the operator $\mathfrak{L}$, defined by (200), which is easily demonstrated. Thus, using the equation for h_1, the extremal value is also given by

$$S_{12} - P_{12} - P_{21} = S_{21} - P_{12} - P_{21} = -P_{12} \tag{230}$$

and so

$$P_{21} = P_{12}. \tag{231}$$

The relation between the P_{ij} and the λ_{ij}, found by substituting (219) into (201) and (202), and comparing with (203) and (204), is

$$P_{ij} = -(n_e T_e \tau_{ei}/m_e)\lambda_{ij} \tag{232}$$

and thus

$$\lambda_{12} = \lambda_{21}. \tag{233}$$

These variational principles are directly related to the rate of entropy production (37). By retaining only terms linear in the expansion parameter ϵ in (37), and using the small mass-ratio approximation for the electron–ion collision term (121), the electron entropy production rate is found to be given by

$$\dot{S}_e = -\int d^3v\, \hat{g}_{e1}\left[C^\ell_{ee} g_{e1} + \nu_{ei}\mathfrak{L} g_{e1}\right] \tag{234}$$

where g_{e1} is defined by (195); it is positive-definite:

$$\dot{S}_e \geq 0 \tag{235}$$

with equality only if $g_{e1} = 0$. By substituting (219) this becomes

$$\dot{S}_e = -\sum_{ij} S_{ij} A_i A_j \geq 0 \tag{236}$$

where S_{ij} is defined by (223). Since the extremal values of the S_{ij} are given by $S_{ij} = P_{ij}$, using also (232),

$$\dot{S}_e = (p_e \tau_{ei}/m_e)\sum_{ij}\lambda_{ij} A_i A_j \geq 0. \tag{237}$$

Thus, the matrix of transport coefficients λ_{ij} is both positive-definite and symmetric. It follows that the diagonal terms and the determinant are positive:

$$\lambda_{11} > 0, \qquad \lambda_{22} > 0, \qquad \lambda_{11}\lambda_{22} - \lambda_{12}^2 > 0. \tag{238}$$

Transport in a plasma with multiple ion species

Transport in a plasma with an arbitrary number of ion species, plus electrons, is now considered. More than one ion species may be present in a plasma because of the presence of impurities, or because there is intentionally a mixture of ion species, as in a fusion reactor. The ratios of the masses of every pair of ion species, m_a/m_b, will be new parameters in the problem, which will be allowed to have any value. All of the ion species will be assumed to have the same temperature, $T_a = T$, but the electron temperature may be different: $T_e \neq T$. General formulas were given above for perpendicular transport in a strong magnetic field. Thus, only parallel transport need be considered.

Electron parallel transport. The transport problem for the electrons can be reduced to that for a simple plasma. By using (121) with the definition

$$\nu_{ea} = \frac{n_a z_a^2}{\sum_c n_c z_c^2}\nu_{ei} \tag{239}$$

where

$$\nu_{ei} = (3\pi^{1/2}/4)\tau_{ei}^{-1}(v_e/v)^3 \tag{240}$$

with

$$\tau_{ei}^{-1} = \frac{4}{3\pi^{1/2}}\frac{n_e \Gamma_e z_{eff}}{v_e^3} \tag{241}$$

and

$$z_{eff} = \sum_c n_c z_c^2 \Big/ \sum_c n_c z_c, \tag{242}$$

$$C^\ell_{ee} f_{e1} + \nu_{ei}\left(\mathfrak{L} f_{e1} + \frac{2v_\| u_{eff}}{v_e^2} f_{e0}\right) = v_\|\left[\left(\frac{\nabla_\| p_e}{p_e} + \frac{e}{T_e}E_\|\right) + \left(\frac{v^2}{v_e^2} - \frac{5}{2}\right)\frac{\nabla_\| T_e}{T_e}\right] f_{e0} \tag{243}$$

where

$$u_{\mathrm{eff}} = \sum_c n_c z_c^2 u_{c\|} \Big/ \sum_c n_c z_c^2. \tag{244}$$

Note that

$$\sum_c n_c z_c = n_e \tag{245}$$

by charge neutrality. By making the substitution

$$f_{e1} = \left(2 v_\| u_{\mathrm{eff}} / v_e^2\right) f_{e0} + g_{e1} \tag{246}$$

(198) is again obtained for g_{e1}, with ν_{ei} now defined by (240). Thus, the electron current density

$$j_{e\|} = -e \int \mathrm{d}^3 v\, v_\| g_{e1}$$

and the electron heat flux

$$q_{e\|} = T_e \int \mathrm{d}^3 v\, v_\| \left(\frac{v^2}{v_e^2} - \frac{5}{2} \right) g_{e1}$$

are given by the transport relations (203) and (204) in which τ_{ei} is given by (241).

Note that the total friction force is given, from (194), by

$$F_{e\|} = \int \mathrm{d}^3 v\, m_e v_\| \nu_{ei} \left(\mathfrak{L} f_{e1} + \frac{2 v_\| u_{\mathrm{eff}}}{v_e^2} f_{e0} \right) = \int \mathrm{d}^3 v\, m_e v_\| \nu_{ei} \mathfrak{L} g_{e1}. \tag{247}$$

It is given in terms of the electron current density and the electron heat flux by the inverse transport relation (207), in which τ_{ei} is given by (241). Note that $j_{e\|}$ is not equal to the total current density:

$$j_\| = -e \int \mathrm{d}^3 v\, v_\| f_{e1} + e \sum_c n_c z_c u_{c\|}. \tag{248}$$

The ion contribution is

$$j_{i\|} = j_\| - j_{e\|} = e \sum_c n_c z_c \left(u_{c\|} - u_{\mathrm{eff}} \right) \tag{249}$$

where u_{eff} is defined by (244).

Ion parallel transport. The ion transport problem can be treated in a fairly general way also. For ion species a the first-order kinetic equation is

$$\sum_c \left[C_{ac}(f_{a1}, f_{c0}) + C_{ac}(f_{a0}, f_{c1}) \right] + C_{ae}^{\ell} = v_\| \left[\left(\frac{\nabla_\| p_a}{p_a} - \frac{e_a}{T} E_\| \right) + \left(\frac{v^2}{v_a^2} - \frac{5}{2} \right) \frac{\nabla_\| T}{T} \right] f_{a0}. \tag{250}$$

Using the same reasoning as in the section on parallel transport, the ion–electron collision terms may be approximated by

$$C_{ae}^{\ell} \simeq -\left(F_{ea\|}/p_a \right) v_\| f_{a0} = \left(F_{ae\|}/p_a \right) v_\| f_{a0}. \tag{251}$$

From (152), the pressure gradient and electric field terms are given by

$$\frac{\nabla_\| p_a}{p_a} - \frac{e_a}{T} E_\| = \left(\sum_b F_{ab\|} + F_{ae\|} \right) \Big/ p_a. \tag{252}$$

By making this replacement in (250), the terms proportional to $F_{ae\|}$ cancel, and the equation becomes

$$\sum_c \left[C_{ac}(f_{a1}, f_{c0}) + C_{ac}(f_{a0}, f_{c1}) \right] = v_\| \left[\frac{F_a}{p_a} + \left(\frac{v^2}{v_a^2} - \frac{5}{2} \right) \frac{\nabla_\| T}{T} \right] f_{a0} \tag{253}$$

where

$$F_a = \sum_c F_{ac\|}. \tag{254}$$

Multiplying (253) by $m_a v_\| (v^2/v_a^2 - \frac{5}{2})$ and integrating over velocity gives for the parallel heat friction for species a:

$$G_a \equiv \int \mathrm{d}^3 v\, m_a v_\| \left(\frac{v^2}{v_a^2} - \frac{5}{2} \right) \sum_c \left[C_{ac}(f_{a1}, f_{c0}) + C_{ac}(f_{a0}, f_{c1}) \right] = \tfrac{5}{2} n_a \nabla_\| T. \tag{255}$$

F_a and G_a will first be considered to be given quantities, so the the $u_{a\|}$ and $q_{a\|}$ are to be determined in terms of them. An approximate solution of (253) (which consists of one equation for each ion species) can be found with the help of the variational principle

$$\delta(S - 2P) = 0$$

where

$$S = \sum_{a,b} \int \mathrm{d}^3 v\, \hat{f}_{a1} \left[C_{ab}(f_{a1}, f_{b0}) + C_{ab}(f_{a0}, f_{b1}) \right] \tag{256}$$

and

$$P = \frac{1}{T} \sum_a \int \mathrm{d}^3 v \frac{f_{a1}}{n_a} v_\| \left[F_a + \frac{2}{5} \left(\frac{v^2}{v_a^2} - \frac{5}{2} \right) G_a \right]. \tag{257}$$

The distribution functions will be expanded in Sonine polynomials (Braginskii, 1965):

$$\hat{f}_{a1} = \frac{2 v_\|}{v_a} \sum_{k=0}^{N} u_{ak} L_k^{(3/2)}\left(v^2/v_a^2 \right) \tag{258}$$

where

$$L_0^{(3/2)}(x)=1, \qquad L_1^{(3/2)}(x)=\tfrac{5}{2}-x,$$

$$L_2^{(3/2)}(x)=\frac{35}{8}-\frac{7}{2}x+\frac{1}{2}x^2, \tag{259}$$

and so on. An exact solution would be obtained in the limit $N\to\infty$; it is found that sufficient accuracy is obtained by taking $N=2$. Note that the $k=0$ and $k=1$ coefficients are proportional to the mean velocity and the heat flux:

$$u_{a0}\equiv u_{a\parallel}/v_a, \qquad u_{a1}\equiv -\tfrac{2}{5}q_{a\parallel}/v_a p_a. \tag{260}$$

With the trial function given by (258),

$$P=\frac{1}{T}\sum_a\left[u_{a\parallel}F_a+\tfrac{2}{5}(q_{a\parallel}/p_a)G_a\right] \tag{261}$$

using the orthogonality of the Sonine polynomials, and

$$S=\frac{8}{3\pi^{1/2}}\sum_{a,b}n_a\nu_{ab}\sum_{k,j}u_{ak}\left(M_{ab}^{kj}u_{aj}+N_{ab}^{kj}u_{bj}\right) \tag{262}$$

where

$$\nu_{ab}\equiv\frac{n_b z_b^2\Gamma_a}{v_a^3}=\frac{4\pi n_b z_a^2 z_b^2 e^4\ln\Lambda}{m_a^2 v_a^3}. \tag{263}$$

The M_{ab}^{kj} and N_{ab}^{kj} are dimensionless matrix elements of the differential and integral collision operators, respectively. That is,

$$n_a\nu_{ab}M_{ab}^{kj}=\frac{3\pi^{1/2}}{4}\int d^3v\frac{v_\parallel}{v_a}L_k^{(3/2)}(v^2/v_a^2)C_{ab}\left(\frac{2v_\parallel f_{a0}}{v_a}L_j^{(3/2)}(v^2/v_a^2),f_{b0}\right) \tag{264}$$

and

$$n_a\nu_{ab}N_{ab}^{kj}=\frac{3\pi^{1/2}}{4}\int d^3v\frac{v_\parallel}{v_a}L_k^{(3/2)}(v^2/v_a^2)C_{ab}\left(f_{a0},\frac{2v_\parallel f_{b0}}{v_b}L_j^{(3/2)}(v^2/v_b^2)\right). \tag{265}$$

Because the differential and integral collision operators are separately self-adjoint, in the sense given by (140) and (141), when $T_a=T_b$, these matrix elements have the symmetry properties

$$M_{ab}^{kj}=M_{ab}^{jk} \tag{266}$$

$$n_a\nu_{ab}N_{ab}^{kj}=n_b\nu_{ba}N_{ba}^{jk} \tag{267}$$

or

$$N_{ab}^{kj}=(m_a/m_b)^{1/2}N_{ba}^{jk}. \tag{268}$$

Using momentum conservation (25) with $f_{b1}=0$, the following additional relation between matrix elements is found:

$$M_{ab}^{0k}+N_{ba}^{0k}=0. \tag{269}$$

Explicit expressions for these matrix elements for $k,j\leqslant 2$ are as follows (Hirshman, 1977).

$$M_{ab}^{00}=-\frac{(1+m_a/m_b)}{(1+v_b^2/v_a^2)^{3/2}}=-N_{ba}^{00},\ M_{ab}^{01}=-\frac{3}{2}\frac{(1+m_a/m_b)}{(1+v_b^2/v_a^2)^{5/2}}=-N_{ba}^{01}, \tag{270}$$

$$M_{ab}^{02}=-\frac{15}{8}\frac{(1+m_a/m_b)}{(1+v_b^2/v_a^2)^{7/2}}=-N_{ba}^{02},$$

$$M_{ab}^{11}=-\frac{\left[\frac{15}{2}(v_b/v_a)^4+4(v_b/v_a)^2+\frac{13}{4}\right]}{(1+v_b^2/v_a^2)^{5/2}},$$

$$M_{ab}^{12}=-\frac{\left[\frac{63}{4}(v_b/v_a)^4+6(v_b/v_a)^2+\frac{69}{16}\right]}{(1+v_b^2/v_a^2)^{7/2}} \tag{271}$$

$$M_{ab}^{22}=-\frac{\left[\frac{175}{8}(v_b/v_a)^8+28(v_b/v_a)^6+\frac{459}{8}(v_b/v_a)^4+17(v_b/v_a)^2+\frac{433}{64}\right]}{(1+v_b^2/v_a^2)^{9/2}}, \tag{272}$$

$$N_{ab}^{11}=\frac{27}{4}\frac{T_a}{T_b}\frac{(v_b/v_a)^3}{(1+v_b^2/v_a^2)^{5/2}},\qquad N_{ab}^{12}=\frac{225}{16}\frac{T_a}{T_b}\frac{(v_b/v_a)^5}{(1+v_b^2/v_a^2)^{7/2}}, \tag{273}$$

$$N_{ab}^{22}=\frac{2625}{64}\frac{T_a}{T_b}\frac{(v_b/v_a)^5}{(1+v_b^2/v_a^2)^{9/2}}. \tag{274}$$

Note that in the present transport theory application of these matrix elements $T_a=T_b$ must be taken, so that $v_b^2/v_a^2=m_a/m_b$. In fact, only if $T_a=T_b$ can the symmetry relations (266) and (268) be used to supply the remaining expressions for $k,j\leqslant 2$. Note that the M_{ab}^{kj} and N_{ab}^{kj} do not depend on n_a, n_b, z_a or z_b; they depend only on the mass ratio m_a/m_b and the ratio of the thermal velocities, v_b/v_a.

$S-2P$ is minimized with respect to the variational parameters u_{ak}, by using

$$(\partial/\partial u_{ak})(S-2P)=0 \tag{275}$$

which yields

$$\frac{8}{3\pi^{1/2}}\sum_b n_a\nu_{ab}\sum_{j=0}^{N}\left(M_{ab}^{kj}u_{aj}+N_{ab}^{kj}u_{bj}\right)=\frac{v_a}{T}(F_a\delta_{k0}-G_a\delta_{k1}). \tag{276}$$

These linear equations for the coefficients u_{ak} are the same ones which are obtained using the moment method (Hirshman, 1977). Using (276) to simplify (262) shows that $S=P$ (identically), so that the variational expression becomes $S-2P=-P$ for any number $(N+1)$ of terms in the trial function.

It is now convenient to regard the parameters $u_{a\parallel}$, $q_{a\parallel}$ as given and to find expressions for the friction force F_a and heat friction G_a in terms of them. Thus, (276) needs to be solved for the u_{ak} for $k\geqslant 2$, in terms of the u_{a0} and u_{a1}. Then, substitution of these back into (276) for $k=0$ and $k=1$ will yield relations of the form

$$F_a = \sum_b \left(l_{11}^{ab} u_{\parallel b} + \frac{2}{5} l_{12}^{ab} \frac{q_{\parallel b}}{p_b} \right) \tag{277}$$

$$G_a = \sum_b \left(l_{21}^{ab} u_{\parallel b} + \frac{2}{5} l_{22}^{ab} \frac{q_{\parallel b}}{p_b} \right). \tag{278}$$

By substituting these into (261) and using $S-2P=-P$ it becomes clear that the l_{ij}^{ab} obtained by solving the linear equations (276) in the manner described above are variational. Therefore, good accuracy is expected with only a few terms in the trial functions.

By choosing $N=2$ in (258), that is three terms in the trial functions, only the $k=2$ version of (276) needs to be solved:

$$n_a z_a^2 \sum_b n_b z_b^2 \sum_{j=0}^{2} \left(M_{ab}^{2j} u_{aj} + N_{ab}^{2j} u_{bj} \right) = 0. \tag{279}$$

These equations, one for each ion species a, are to be solved for the same number of unknowns, the u_{a2}. The solution can be carried out using the method of Boley et al. (1979), as follows.

The summation over ion species indices b can be written as the sum over different isotopes (i.e. different ion masses m_b) of the sums over the different ionization states (i.e. charges $z_b e$) for a given isotope:

$$\sum_b = \sum_{m_b} \sum_{i_b} .$$

The species index b is equivalent to the pair of indices (m_b, i_b), in what follows. Summing on the ionization state index i_a in (279), over all states corresponding to a given mass m_a, and dividing by $\Sigma_{i_a} n_a z_a^2$ gives:

$$\sum_{j=0}^{2} \left\{ \left[\sum_{m_b} \left(\sum_{i_b} n_b z_b^2 \right) M_{ab}^{2j} \right] \bar{u}_{aj} + \sum_{m_b} \left(\sum_{i_b} n_b z_b^2 \right) N_{ab}^{2j} \bar{u}_{bj} \right\} = 0. \tag{280}$$

The averages

$$\bar{u}_{ak} = \sum_{i_a} n_a z_a^2 u_{ak} \Big/ \sum_{i_a} n_a z_a^2 \tag{281}$$

have been defined, which depend only on the indices m_a, k, and the fact that M_{ab}^{kj} and N_{ab}^{kj} depend only on the mass indices m_a, m_b has been used. Further, with the definition

$$L_{ab}^{kj} = \left[\sum_{m_c} \left(\sum_{i_c} n_c z_c^2 \right) M_{ac}^{kj} \right] \delta_{ab} + \left(\sum_{i_b} n_b z_b^2 \right) N_{ab}^{kj}, \tag{282}$$

(280) can be written as

$$\sum_{m_b} L_{ab}^{22} \bar{u}_{b2} = - \sum_{j=0}^{1} \sum_{m_b} L_{ab}^{2j} \bar{u}_{bj} . \tag{283}$$

The solution can then be written in terms of the inverse of the matrix L^{22}:

$$\bar{u}_{a2} = - \sum_{j=0}^{1} \sum_{m_d} \sum_{m_b} (L^{22})_{ad}^{-1} L_{db}^{2j} \bar{u}_{bj} . \tag{284}$$

Since usually only a small number of different isotopes are of interest, the dimension of the matrices L^{20}, L^{21}, L^{22} can be considered to be small.

The number of possible ionization states for all of the isotopes is generally much larger than the number of isotopes. However, the unknowns u_{a2} can be determined in terms of the averages $\bar{u}_{a2}$ in an even simpler way. For, by dividing (279) by $n_a z_a^2$ and subtracting (280), the terms containing the N_{ab}^{2j} cancel, and

$$\sum_{j=0}^{2} \left[\sum_{m_b} \left(\sum_{i_b} n_b z_b^2 \right) M_{ab}^{2j} \right] (u_{aj} - \bar{u}_{aj}) = 0. \tag{285}$$

Solving for u_{a2} gives

$$u_{a2} = \bar{u}_{a2} - \left[\sum_{j=0}^{1} M_a^{2j} (u_{aj} - \bar{u}_{aj}) \right] \Big/ M_a^{22} \tag{286}$$

where

$$M_a^{kj} \equiv \sum_{m_b} \left(\sum_{i_b} n_b z_b^2 \right) M_{ab}^{kj} . \tag{287}$$

Thus, using (286) and (284), the u_{a2} are determined in terms of the u_{a0} and u_{a1} and substitution into (276) with $k=0$ and $k=1$ finally gives the inverse transport relations of the form of (277) and (278). It is more convenient to express these in the form

$$F_a = \frac{4}{3\pi^{1/2}} \frac{m_a n_a \Gamma_a}{v_a^2} \sum_{k=0}^{1} \left(\bar{\mu}_{0k}^a (u_{ak} - \bar{u}_{ak}) + \sum_{m_b} \bar{\mu}_{0k}^{ab} \bar{u}_{bk} \right) \tag{288}$$

$$G_a = - \frac{4}{3\pi^{1/2}} \frac{m_a n_a \Gamma_a}{v_a^2} \sum_{k=0}^{1} \left(\bar{\mu}_{1k}^a (u_{ak} - \bar{u}_{ak}) + \sum_{m_b} \bar{\mu}_{1k}^{ab} \bar{u}_{bk} \right) \tag{289}$$

where, using the results expressed by (284) and (286),

$$\bar{\mu}_{jk}^a = M_a^{jk} - M_a^{j2} M_a^{2k} / M_a^{22} \tag{290}$$

and

$$\bar{\mu}_{jk}^{ab} = L_{ab}^{jk} - \sum_{m_c} \sum_{m_d} L_{ac}^{j2} (L^{22})_{cd}^{-1} L_{db}^{2k} \tag{291}$$

where

$$u_{a0} \equiv u_{a\parallel}/v_a, \qquad u_{a1} \equiv -\tfrac{2}{5}(q_{a\parallel}/v_a p_a) \tag{292}$$

and where the averages indicated by the bars are defined by (281).

These relations are best evaluated by a computer, for any given problem. Since only function evaluation and the solution of a small set of linear equations is involved, the evaluation is not time-consuming. The inversion of (288) and (289), giving the mean velocities and the heat fluxes in terms of the friction forces and heat friction vectors, or using (252) and (255), in terms of the pressure and temperature gradients, i.e. the transport relations, can also be done by using a computer.

In some applications, it is the inverse transport relations in the form given by (288) and (289) which are needed. For example, in Pfirsch–Schluter transport in a toroidal confinement system (Hazeltine and Hinton, 1973; Hirshman, 1977), the parallel flows $u_{a\parallel}, q_{a\parallel}$ are given, as a consequence of the geometrical and timescale constraints, in terms of radial gradients, while the radial particle and heat fluxes are determined by the parallel friction and heat friction. Thus, (288) and (289) lead directly to radial transport relations for this application.

Moment equations

To summarize the results of the transport theory, the moment equations for a multiple ion species plasma are now presented. In the following, subscripts a and b refer to ionic species; electron quantities will be denoted explicitly by "e". The conservation of particles is expressed by

$$\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e \boldsymbol{u}_e) = 0 \tag{293}$$

$$\frac{\partial n_a}{\partial t} + \nabla\cdot(n_a \boldsymbol{u}_a) = n_e \sum_{i_b} S_{ab} n_b \tag{294}$$

where the ion sources due to ionization and recombination have been included, with S_{ab} denoting the rate coefficients. A discussion of these coefficients is beyond the scope of this article; all other such atomic effects, such as electron energy loss due to ionization, will henceforth be omitted.

The ion version of the energy equation is, after summation over all ion species,

$$\frac{\partial}{\partial t}\left(\frac{3}{2} nT\right) + \nabla\cdot\boldsymbol{Q} = \sum_a n_a e_a \boldsymbol{u}_{a1}\cdot\boldsymbol{E} + \sum_a (Q_{ae} + \boldsymbol{u}_{a1}\cdot\boldsymbol{F}_{ae}) \tag{295}$$

where $n = \sum_a n_a$ is the total ion density. T is the common ion temperature, and

$$\boldsymbol{Q} = \sum_a \left(\boldsymbol{q}_{a1} + \frac{5}{2} n_a T \boldsymbol{u}_{a1}\right) \tag{296}$$

is the total ion energy flux. The total electron–ion energy transfer rate is

$$Q_{ie} = \sum_a Q_{ae} = \frac{3 m_e n_e}{\tau_{ei}} \frac{\sum_b n_b z_b^2/m_b}{n_e z_{eff}} (T_e - T) \tag{297}$$

where τ_{ei} and z_{eff} are defined by (241) and (242), respectively. The remaining terms on the right-hand side can be transformed by using

$$\boldsymbol{u}_{a1}\cdot\left(\boldsymbol{F}_{ae} + \sum_b \boldsymbol{F}_{ab}\right) = \boldsymbol{u}_{a1}\cdot(\nabla p_a - n_a e_a \boldsymbol{E}) \tag{298}$$

which follows from the fact that $\boldsymbol{u}_{a1}$ satisfies the equation

$$-\nabla p_a + n_a e_a\left(\boldsymbol{E} + \frac{\boldsymbol{u}_{a1}}{c} \times \boldsymbol{B}\right) + \boldsymbol{F}_{ae} + \sum_b \boldsymbol{F}_{ab} = 0. \tag{299}$$

The terms explicitly proportional to the electric field thus cancel, leaving

$$\frac{\partial}{\partial t}\left(\frac{3}{2} nT\right) + \nabla\cdot\boldsymbol{Q} = Q_{ie} + \sum_a \boldsymbol{u}_{a1}\cdot\left(\nabla p_a - \sum_b \boldsymbol{F}_{ab}\right). \tag{300}$$

The terms on the right-hand side of the electron energy equation which are due to collisions with the ions are just the negatives of the corresponding terms in the ion energy equation. Thus, the electron energy equation is

$$\frac{\partial}{\partial t}\left(\frac{3}{2} n_e T_e\right) + \nabla\cdot\boldsymbol{Q}_e = \boldsymbol{j}\cdot\boldsymbol{E} - Q_{ie} - \sum_a \boldsymbol{u}_{a1}\cdot\left(\nabla p_a - \sum_b \boldsymbol{F}_{ab}\right) \tag{301}$$

where

$$\boldsymbol{Q}_e = \boldsymbol{q}_{e1} + \tfrac{5}{2} n_e T_e \boldsymbol{u}_{e1} \tag{302}$$

where the terms explicitly proportional to the electric field have added to give the total joule heating rate.

The fluxes which appear in these equations are the sums of contributions which are perpendicular and parallel to the magnetic field:

$$n_a \boldsymbol{u}_a = n_a \boldsymbol{u}_{a\perp} + \hat{\boldsymbol{n}} n_a u_{a\parallel} \tag{303}$$

$$\boldsymbol{q}_a = \boldsymbol{q}_{a\perp} + \hat{\boldsymbol{n}} q_{a\parallel} \tag{304}$$

where $\hat{\boldsymbol{n}}$ is a unit vector tangent to the magnetic field lines.

The perpendicular ion fluxes are given by

$$n_a \boldsymbol{u}_{a\perp} = n_a \frac{c}{B^2} \boldsymbol{E} \times \boldsymbol{B} + \frac{c}{e_a B^2} \boldsymbol{B} \times \nabla p_a + (n_a \boldsymbol{u}_{a\perp})_i + (n_a \boldsymbol{u}_{a\perp})_e \tag{305}$$

where $(n_a \boldsymbol{u}_{a\perp})_i$ is given by (177) with the electron term omitted. The electron collisional contribution is given in terms of the perpendicular electron–ion friction from (176):

$$(n_a \boldsymbol{u}_{a\perp})_e \equiv -\frac{c}{e_a B^2} \boldsymbol{B} \times \boldsymbol{F}_{ae} \tag{306}$$

$$= \frac{n_e}{z_a m_e \Omega_e^2 \tau_{ea}} \left(-\frac{\nabla_\perp p_a}{z_a n_a} - \frac{\nabla_\perp p_e}{n_e} + \frac{3}{2} \nabla_\perp T_e\right).$$

The perpendicular electron flux is

$$n_e u_{e\perp} = n_e \frac{c}{B^2} E \times B - \frac{c}{eB^2} B \times \nabla p_e - \frac{n_e}{m_e \Omega_e^2 \tau_{ei}} \left(\frac{\nabla_\perp p_e}{n_e} + \frac{\sum_b z_b \nabla_\perp p_b}{n_e z_{eff}} - \frac{3}{2} \nabla_\perp T_e \right). \qquad (307)$$

The perpendicular ion heat flux is given by

$$q_{a\perp} = \frac{5}{2} \frac{c p_a}{e_a B^2} B \times \nabla T_a + q_{a\perp}^{(1)} \qquad (308)$$

with $q_{a\perp}^{(1)}$ given by (178), where the electron term can be neglected. The perpendicular electron heat flux is

$$q_{e\perp} = -\frac{5}{2} \frac{c p_e}{eB^2} B \times \nabla T_e + \frac{p_e}{m_e \Omega_e^2 \tau_{ei}} \left[\frac{3}{2} \left(\frac{\nabla_\perp p_e}{n_e} + \frac{\sum_b z_b \nabla_\perp p_b}{n_e z_{eff}} \right) - \left(\frac{13}{4} + \frac{\sqrt{2}}{z_{eff}} \right) \nabla_\perp T_e \right]. \qquad (309)$$

The parallel electron flux is given by

$$n_e u_{e\parallel} = -j_{\parallel e}/e + n_e u_{eff} \qquad (310)$$

where u_{eff} is defined by (244) and $j_{\parallel e}$ is given by the transport relation (203). The parallel electron heat flux is given by the transport relation (204). In both (203) and (204), τ_{ei} is to be defined by (241).

The parallel ion fluxes and the parallel ion heat fluxes can be obtained by inverting the inverse transport relations given by (288) and (289). Equations (254), (252), and (255) are then used for F_a and G_a. Note that in (252) the electron friction term, $F_{ae\parallel}$, is given by

$$F_{ae\parallel} = -F_{ea\parallel}$$

where, using (239), (247) and (246),

$$F_{ea\parallel} = \frac{n_a z_a^2}{(\sum_c n_c z_c^2)} \left[F_{e\parallel} + \left(\frac{m_e n_e}{\tau_{ei}} \right) (u_{a\parallel} - u_{eff}) \right] \qquad (311)$$

with $F_{e\parallel}$ given by (207).

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